211. Generalizations of the Stone-Weierstrass Approximation Theorem*)

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The celebrated Stone-Weierstrass theorem for the continuous functions on compact Hausdorff spaces has been extended to those on more general spaces [1], [3], [4], [8]. The purpose of the present note is to present some generalizations of the theorem and the Stone-Tietze extension theorem to the vector-valued continuous functions on completely regular spaces.

Let X be a completely regular space, $C(X, K)$ the algebra of all complex continuous functions (bounded or unbounded) on X and $\mathfrak{M}(C(X, K))$ the maximal ideal space of $C(X, K)$. We recall two results proved in [10], [11]: (1) $\mathfrak{M}(C(X, K))$ endowed with Stone topology (hull-kernel) is homeomorphic to the Stone-Čech compactification βX and (2) each $f \in C(X, R)$ can be extended to a continuous function $\tilde{f}$ over βX with values in $[-\infty, \infty]$. The set of all $\tilde{f}$ for $f \in C(X, K)$ is denoted by $\tilde{C}(X, K)$.

Definition 1. Let X be a completely regular space and S a subset of $C(X, K)$. A function $f \in C(X, K)$ is said to be a limit point of S under uniform topology if f can be uniformly approximated by the functions in S on subsets of X on which f is bounded.

Lemma 1. *Let X be a completely regular space and $C(X, R)$ the algebra of all real continuous functions on X. If a subalgebra S of $C(X, R)$ contains the identity element and separates $\mathfrak{M}(C(X, R))$, then S is dense in $C(X, R)$ under uniform topology. The same result holds for $C(X, R)$ if S is selfadjoint.*

Proof. By the classical Weierstrass theorem ([9], p. 175) there exists a polynomial $P_n(t)$ such that $||t| - P_n(t)| < 1/n$ for $t \in [-n, n]$. Then $||f(x)| - P_n(f(x))| < 1/n$ if $|f(x)| \leq n$ and $f \in S$ implies $|f| \in \bar{S}$, the closure of S. $\bar{S}$ is therefore a lattice and all $f_m = (f \wedge m) \vee (-m)$ for positive integers m and $f \in \bar{S}$ belongs to $\bar{S}$. It follows that the bounded functions in $\bar{S}$ separates the compact Hausdorff space $\mathfrak{M}(C(X, R))$ and all bounded real continuous functions on X are elements of $\bar{S}$ as a consequence of the Stone-Weierstrass theorem. Since

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