206. Generalized Product and Sum Theorems for Whitehead Torsion

By Hiroshi HOSOKAWA

(Comm. by Kinjirô KUNUGI, M. J. A., Nov. 12, 1968)

1. Introduction. Let K and L be finite CW-complexes and let $f: K \to L$ be a cellular map. If f is a homotopy equivalence, the Whitehead torsion $\tau(f) \in \mathrm{Wh}(\pi)$ is defined, where $\mathrm{Wh}(\pi)$ is the Whitehead group of the fundamental group π of L (for the definitions, see Milnor [2]).

Whitehead has proved in [4] that K and L are of the same simple homotopy type iff there is a homotopy equivalence $f: K \to L$ such that $\tau(f)=0$.

In 1965, Kwun and Szczarba proved two theorems for Whitehead torsion [1]; one is the Sum Theorem, and the other the Product Theorem. The Sum Theorem is stated as follows.

Theorem I. *Let X and Y be finite cell complexes which are the union of subcomplexes $X=X_1 \cup X_2$, $Y=Y_1 \cup Y_2$, and X_0, Y_0 the intersection $X_0=X_1 \cap X_2$, $Y_0=Y_1 \cap Y_2$. Let $f: X \to Y$ be a cellular map and $f|X_i=f_i: X_i \to Y_i$ $(i=0, 1, 2)$. If f_i are homotopy equivalences and X_0 is connected and simply connected, then f is a homotopy equivalence and*

$$\tau(f)=j_{1*}\tau(f_1)+j_{2*}\tau(f_2), \tag{1}$$

where $j_{i*}: \mathrm{Wh}(\pi_1(Y_i)) \to \mathrm{Wh}(\pi_1(Y))$ are induced by the inclusion maps.

In this paper we shall consider the case when X_0 is non-simply connected. Then we obtain the following result which is a generalization of Theorem I.

Theorem I′. *Let X, Y be finite CW-complexes which are the union of subcomplexes $X=X_1 \cup X_2$, $Y=Y_1 \cup Y_2$. Put $X_0=X_1 \cap X_2$, $Y_0=Y_1 \cap Y_2$. Let $f: X \to Y$ be a cellular map and $f_i=f|X_i: X_i \to Y_i$ be homotopy equivalences $(i=0, 1, 2)$. If X_0 is connected, then f is a homotopy equivalence and*

$$\tau(f)=j_{1*}\tau(f_1)+j_{2*}\tau(f_2)-j_{0*}\tau(f_0), \tag{2}$$

where $j_i: Y_i \to Y$ are inclusions.

In particular, if X_0 is simply connected, then $\tau(f_0)=0$ and hence we get formula (1) from formula (2).

Next, the Product Theorem in [1] reads as follows.

Theorem II. *If C is an acyclic based A-complex and C' a based B-complex, then $\tau(C \otimes_z C')=\chi(C')i_*\tau(C)$, where $\chi(C')$ is the Euler*