205. On Generalized Integrals. III

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In the preceding papers [3], we showed that the special $(E.R.)$ integral is defined as a unique and natural extension of integrals (defined as usual) of step functions, using the method of the ranked space. In fact, to do this, we introduced on the set $\mathcal{E}$ of step functions on $[a, b]$ a set of neighbourhoods, denoted by $V(A, \varepsilon; f)$, and a rank so that $\mathcal{E}$ should become a ranked space. In this ranked space $\mathcal{E}$, we see that if $u: \{V_n(f_n)\}$ is a fundamental sequence of neighbourhoods, the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists almost everywhere, and the sequence of integrals $\int_a^b f_n(x)dx$ converges to a finite limit. Moreover, if $u: \{V_n(f_n)\}$ and $v: \{V_n(g_n)\}$ are two fundamental sequences belonging to the same maximal collection u^*, then we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x) \quad \text{a.e.},$$

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \lim_{n\to\infty} \int_a^b g_n(x)dx.$$

Therefore, each maximal collection u^* in $\mathcal{E}$ determines a function and a value which we can associate to this u^*. $J(u^*)$ denotes the function and $I(u^*)$ denotes the value. If we denote, by $\boldsymbol{U}$, the set of all maximal collections u^*, we have $J(u^*) \neq J(v^*)$ for $u^* \in \boldsymbol{U}$ and $v^* \in \boldsymbol{U}$ such that $u^* \neq v^*$. We denoted, by $\boldsymbol{K}$, the set $\{J(u^*); u^* \in \boldsymbol{U}\}$, and for each $f = J(u^*)$, we defined the integral $I(f)$ of f by taking the value $I(u^*)$. Then, $\boldsymbol{K}$ coincides with the set of $(E.R.)$ integrable functions in the special sense (or A-integrable functions) and we have $I(f) = (E.R.)\int_a^b f(x)dx = (A)\int_a^b f(x)dx$. In this paper, we will show that if we reasonably introduce a set of neighbourhoods and a rank on $\boldsymbol{K}$, then the ranked space $\boldsymbol{K}$ is a completion of the ranked space $\mathcal{E}$ (Theorem 3). Moreover, the special $(E.R.)$ integral is the r-continuous extension of integrals of step functions, and it is a r-continuous linear functional on the complete ranked space $\boldsymbol{K}$ (Theorem 4).

In order to introduce the notion of completion in the ranked spaces,[1] we first recall a few basic concepts in the general ranked spaces. Throughout this paper, we suppose that the ranked spaces

1) For the problem of the completion of the ranked spaces, see [1] and [5].