# 199. On a Problem of MacLane

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1. Let $f(z)$ be a non-constant holomorphic function in $\{|z|<1\}$, having asymptotic values at each point of a dense subset on $\{|z|=1\}$. Such a function is said to belong to the class $\mathcal{A}$ (MacLane [1]). MacLane proposed a problem:

*If $f(z)$ and $g(z)$ belong to $\mathcal{A}$, do $f(z)+g(z)$ and $f(z)g(z)$ belong to $\mathcal{A}$?*

Ryan and Barth [2] answered to this negatively, and raised a further question:

*If $f(z)\in\mathcal{A}$ and $b(z)$ is bounded, are $b(z)f(z)\in\mathcal{A}$?* (We suppose, of course, that $b(z)f(z)$ is not a constant.)

In the present note, we will answer to this positively but only partly. That is, we will prove the following

**Theorem A.** *Let $b(z)$ be a function, holomorphic and bounded in $\{|z|<1\}$, having non-zero Fatou limits on $\{|z|=1\}$ except on a set of the first Baire category. Then, if $f(z)\in\mathcal{A}$, we have $b(z)f(z)\in\mathcal{A}$.*

2. For the sake of convenience, we repeat the definitions due to MacLane [1], with slight modifications in notations.

An arc $\Gamma: z=z(t)$, $0\leqq t<1$, in $\{|z|<1\}$ is said to be the path ending at a point $\zeta$, $|\zeta|=1$, if $z(t)\to\zeta$ as $t\to 1$. A function $f(z)$ is said to have an asymptotic value $a$ ($a=\infty$ permitted) at $\zeta$, if there exists a path $\Gamma$ ending at $\zeta$ on which $f(z)$ has the limit $a$, i.e., if $f(z(t))\to a$ as $t\to 1$. The set of these points is denoted by $A_f(a)$. That is, $A_f(a)$ is the set at each point of which $f(z)$ has the asymptotic value $a$. We put

$$A_f^*=\bigcup_{a\neq\infty}A_f(a),\qquad A_f=A_f^*\cup A_f(\infty).$$

A function $f(z)$ is defined to belong to the class $\mathcal{A}$ if $f(z)$ is holomorphic and non-constant in $\{|z|<1\}$ and the set $A_f$ is dense on $\{|z|=1\}$.

Next we define the sets $B_f^*$ and $B_f$. A point $\zeta$, $|\zeta|=1$, belongs to $B_f^*$ if and only if there exists a path $\Gamma$ ending at $\zeta$, on which $f(z)$ is bounded by some finite constant $M$. The bound $M$ may vary as $\zeta$ and $\Gamma$ vary. We put

$$B_f=B_f^*\cup A_f(\infty).$$

$f(z)$ is defined to belong to the class $\mathcal{B}$ if $f(z)$ is holomorphic and non-constant in $\{|z|<1\}$ and the set $B_f$ is dense on $\{|z|=1\}$.

The set $\{z\,;\,|f(z)|=\lambda\}$, where $\lambda\geqq 0$ is a constant, is called *level set*