226. Relations between Unitary ρ-Dilatations and Two Norms. II

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1. Following [1] [4] [7] an operator T on a Hilbert space H possesses a unitary ρ-dilatation if there exist a Hilbert space K containing H as a subspace, a positive constant ρ and a unitary operator U on K satisfying the following representation

$$T^n = \rho \cdot PU^n \quad (n=1, 2, \cdots\cdots) \tag{1}$$

where P is the orthogonal projection of K on H. Put $\mathcal{C}_\rho$ the class of all operators on H having a unitary ρ-dilatation on a suitable enlarged space K. These classes $\mathcal{C}_\rho$ $(\rho \geqq 0)$ were introduced by Sz-Nagy and C. Foias [7]. They have shown a characterization and the monotonity of $\mathcal{C}_\rho$. In the previous paper [4] we obtained the condition for the operator norm $\|T\|$ and the numerical radius $\|T\|_N$ satisfied by T in $\mathcal{C}_\rho$ $(\rho \leqq 2)$,

that is if $T \in \mathcal{C}_\rho$ $(0 \leqq \rho \leqq 1)$, *then*

$$1/2\|T\| \leqq \|T\|_N \leqq \begin{cases} \|T\| & \left(0 \leqq \|T\| \leqq \dfrac{\rho}{2-\rho}\right) \\ \dfrac{\rho}{2-\rho} & \left(\dfrac{\rho}{2-\rho} \leqq \|T\| \leqq \rho\right) \end{cases}$$

and if $T \in \mathcal{C}_\rho$ $(1 \leqq \rho \leqq 2)$, *then*

$$1/2\|T\| \leqq \|T\|_N \leqq \begin{cases} \|T\| & (0 \leqq \|T\| \leqq 1) \\ 1 & (1 \leqq \|T\| \leqq \rho). \end{cases}$$

In this paper we continue the investigation for classes $\mathcal{C}_\rho$ $(\rho \geqq 2)$. We give a simple necessary condition for $T \in \mathcal{C}_\rho$ $(\rho \geqq 2)$ related to both $\|T\|$ and $\|T\|_N$ and its graphic representation.

2. The following theorems are known and we cite for the sake of convenience ([2] [4] [7]).

Theorem A. *An operator T in H belongs to the class $\mathcal{C}_\rho$ if and only if it satisfies the following conditions*

$$\text{(i)} \begin{cases} (\mathrm{I}_\rho) & \|h\|^2 - 2\left(1-\dfrac{1}{\rho}\right)\mathrm{Re}(zTh, h) + \left(1-\dfrac{2}{\rho}\right)\|zTh\|^2 \geqq 0 \\ & \textit{for } h \textit{ in } H \textit{ and } |z| \leqq 1, \\ (\mathrm{II}) & \textit{the spectrum of } T \textit{ lies in the closed unit disk.} \end{cases}$$

(ii) *If $\rho \leqq 2$, then the condition (I_ρ) implies (II).*

Using the notion of shell, Ch. Davis [2] has proved the following proposition.