10. On Weak Convergence of Transformations in Topological Measure Spaces

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1. Introduction. A sequence $\{T_n\}$ of invertible measure-preserving transformations in the unit interval [0, 1] is said to be convergent weakly to the invertible measure-preserving transformation T if $\lim_{n\to\infty}\|f\circ T_n - f\circ T\| = 0$ for every integrable function f, with $\|\cdot\|$ denoting L^1-norm. It is well-known that (α) and (β) in Theorem 1 below are equivalent.

In this paper we prove that if X is a locally compact metrizable space and μ a σ-finite Radon measure on X, then the equivalence between (α) and (β) also holds (Theorem 1). We see that this generalizes a theorem of Papangelou [2, Theorem 2]. Then it will be natural to ask: does the metrizability of X be dropped in Theorem 1 when X is a compact Hausdorff space? Theorem 3 asserts that the answer is negative.

2. An extension of Papangelou's theorems. Let X be a locally compact Hausdorff space and $\mathfrak{B}$ the σ-field generated by the open subsets of X. The members of $\mathfrak{B}$ will be called the Borel subsets of X. Let μ_1 be a measure on $\mathfrak{B}$ such that

(i) $\mu_1(K)$ is finite for every compact subset K of X,

(ii) $\mu_1(V)=\sup\{\mu_1(K)\mid K$ is compact and $K\subset V\}$ for every open subset V of X,

(iii) $\mu_1(A)=\inf\{\mu_1(V)\mid V$ is open and $A\subset V\}$ for every Borel subset A of X.

We denote by μ the outer measure induced by μ_1 and denote by $\mathfrak{M}$ the σ-field of all subsets of X which are μ-measurable. We say μ on $\mathfrak{M}$ a Radon measure on X. A subset E of X which belongs to $\mathfrak{M}$ will be called measurable in X.

We denote by G the group of all invertible μ-measure-preserving transformations in X.

Difinition. The sequence $\{T_n\}$ in G converges to $T\in G$ weakly if $\lim_{n\to\infty}\mu(T_nA+TA)=0$ for every measurable subset A of X with $\mu(A)<\infty$, or equivalently, if $\lim_{n\to\infty}\|f\circ T_n - f\circ T\|=0$ for every $f\in L^1$.

Theorem 1. *Let X be a locally compact metrizable space and μ a σ-finite Radon measure on X. If T, T_n $(n=1, 2, 3, \cdots)$ are in G then*