4. On the Classical Flows with Discrete Spectra

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Introduction. The main purpose of this paper is to consider the problem of classification or isomorphism of classical flows. Namely, in a certain class of classical flows, it is shown that their spectra and the order of differentiability of their eigenfunctions are the complete invariants (Theorem 2, § 3). This is an analogue of the famous theorem due to von Neumann: unitary equivalence of abstract flows implies their metrical equivalence in the case of ergodic abstract flows with discrete spectra.

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§ 1. Preliminaries. In this section we summarize necessary definitions and theorems. For details, refer to [1], [2].

Definition 1. *Classical flow* means the triple (M, μ, φ_t) (or briefly (φ_t)) formed by a C^∞-manifold M, a finite measure on M defined by a positive continuous density (we assume that $\mu(M)=1$) and one-parameter group (φ_t) of diffeomorphisms of M which preserve the measure μ.

Definition 2. Let (M, μ, φ_t) and (N, ν, ψ_t) be classical flows. (M, μ, φ_t) is *C^ρ-isomorphic* to (N, ν, ψ_t) as classical flows, when there exists a C^ρ-diffeomorphism $\iota: M \to N$ such that $\iota\circ\varphi_t=\psi_t\circ\iota$ for all t, and $\iota(\mu)=\nu$. We denote it by:

$$(M, \mu, \varphi_t) \underset{C^\rho}{\cong} (N, \nu, \psi_t).$$

Definition 3. A flow (φ_t) is called *ergodic*, when the condition

$$\mu\{\varphi_t A \ominus A\}=0 \text{ for all } t \text{ implies } \mu(A)=0$$

or $\mu(A)=1$, where $A\ominus B$ denotes the symmetric difference of two sets A and $B: A\ominus B=A\cup B-A\cap B$.

Let (M, μ, φ_t) be a classical flow, then it induces naturally a one-parameter group of unitary operators $\{U_t\}$ on the Hilbert space $\boldsymbol{H}=L^2(M, \mu)$ of complex valued square summable functions defined on M:

$$(U_t f)(x)=f(\varphi_t x), \text{ for } f\in \boldsymbol{H}.$$

By the decomposition theorem of Stone, these U_t have the following spectral resolution: