20. *On Generalized Integrals.* IV

By Shizu NAKANISHI

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The $(E.R.)$ integral proposed by Prof. K. Kunugi in [1], that is, the $(E.R.)$ integral in the special sense, which is defined as an extension of the Lebesgue integral, cannot always integrate the important functions: for example, the function $1/x$ is not $(E.R.)$ integrable in the special sense in $[-1, 1]$. Prof. K. Kunugi remarked in [1] that the method of change of the variable admits the extension of the range of the integration. We see the precise definition in [2]. In fact, to do this, he used the function g defined in $[\alpha, \beta]$, which is non-negative and Lebesgue-integrable. Let $G(x)$ be the indefinite integral of g such that $G(\alpha)=a$ and $G(\beta)=b$. For the function $f(t)$ defined in $[a, b]$, if the function $f_1(x)=f(G(x))g(x)$ is $(E.R.)$ integrable in the special sense in $[\alpha, \beta]$, the function $f(t)$ is said to be $(E.R.)$ integrable in the extended sense in $[a, b]$, and we understand by the integral of $f(t)$ in the extended sense in $[a, b]$ the number $(E.R.) \int_\alpha^\beta f(G(x))g(x)dx$. For example, the function $1/t$ is $(E.R.)$ integrable in the extended sense in $[-1, 1]$. For, if we put $g(x)=1/(|x| \log(1/|x|)^2)$, and put $G(x)=1/\log(1/x)$ for $x>0$, $G(0)=0$, $G(x)=-G(-x)$ for $x<0$, then $G(x)$ is the indefinite integral of $g(x)$ such that $G(-1/e)=-1$ and $G(1/e)=1$, and the function $g(x)/G(x)=1/(x \log(1/|x|))$ is $(E.R.)$ integrable in the special sense in $[-1/e, 1/e]$. Hence, the function $1/t$ is $(E.R.)$ integrable in the extended sense in $[-1, 1]$, and the integral is $(E.R.) \int_{-1/e}^{1/e} 1/(x \log(1/|x|))dx=0$.

This theory of Prof. K. Kunugi has been extended to the abstract measure space in [4] by H. Okano. He termed it $(E.R.)$ integral with respect to a measure ν, or $(E.R.\ \nu)$ integral.

In the preceding papers [3], we obtained the set $\boldsymbol{K}$ of special $(E.R.)$ integrable functions as a completion of the set $\mathcal{E}$ of step functions, and showed that the special $(E.R.)$ integral is a continuous linear functional on the complete ranked space $\boldsymbol{K}$. The purpose of this paper is to define the $(E.R.)$ integral in the extended sense in a similar way. Let $\varphi(t)$ be a positive, Lebesgue-integrable function defined in a finite or infinite interval $[a, b]$[1] and $\Phi(t)$ be the indefinite integral of $\varphi(t)$

1) The infinite interval $[a, b]$ designates one of the intervals $-\infty<x<+\infty$, $a\leq x<+\infty$ $(a\neq-\infty)$ and $-\infty<x\leq b$ $(b\neq+\infty)$.