36. On Regular Duo Rings

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Recently the author proved some characterizations of semigroups, which are semilattices of groups (see [3]-[5]). One of these characterizations reads as follows:

Theorem 1. *A semigroup S is a semilattice of groups if and only if the relation*

(1) $$R \cap L = LR$$

holds for any left ideal L and right ideal R of S.

In this paper we formulate an analogous assertion in the theory of associative rings. A ring A is called a *duo* (or *two-sided*) ring if every one-sided (left or right) ideal of A is a two-sided ideal (see Hille [1]). An associative ring A is said to be *regular* if to every element a in A there exists an element $x \in A$ such that $axa = a$ (see von Neumann [7]). Interesting characterizations of regular rings were obtained by Kovács [2] and Luh [6].

Theorem 2. *An associative ring A is a regular duo ring if and only if the relation* (1) *holds for any left ideal L and right ideal R of A.*

Proof. *Necessity.* Let A be a regular duo ring. Then A satisfies the relation

(2) $$R \cap L = RL$$

for every left ideal L and right ideal R of A (see Kovács [2]). Now (2) implies (1) because each one-sided ideal of a duo ring is a two-sided ideal.

Sufficiency. Let A be an associative ring having the property (1) for any left ideal L and right ideal R of A. In case of $R = A$ the relation (1) implies

(3) $$A \cap L = LA,$$

whence every left ideal L of A is also a right ideal of A. Similarly, in case of $L = A$ (1) implies that

(4) $$R \cap A = AR,$$

whence the right ideal R of A is a two-sided ideal of A. Therefore A is a duo ring. Finally (1) implies (2), which is equivalent with the regularity of A.

Theorem 2 is completely proved.