# *34. Modular Pairs in Atomistic Lattices with the Covering Property*

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**1. Introduction.** In the previous paper [4], a lattice $L$ is called a DAC-lattice when both $L$ and its dual are atomistic lattices with the covering property. The lattice $\mathcal{L}$ of closed subspaces of a linear system, appeared in Mackey [2], is an example of a DAC-lattice. In [2; p. 168], Mackey proved that a pair of elements of $\mathcal{L}$ is both modular and dual-modular if and only if it is stable modular. In this paper we shall show (Theorem 2) that this statement can be proved in general DAC-lattices. As a consequence of this result, we shall obtain a condition on a DAC-lattice which is equivalent to cross-symmetry. In the last section, we shall show some results on cross-symmetry of the lattice of closed subspaces of a locally convex space.

**2. Symmetry of modular relations.** Let $a$ and $b$ be elements of a lattice. We say that $(a, b)$ is a *modular pair* (resp. a *dual-modular pair*) and write $(a, b)M$ (resp. $(a, b)M^*$) when

$$(c\vee a)\wedge b=c\vee(a\wedge b) \quad \text{for every} \quad c\leqq b$$

(resp. $$(c\wedge a)\vee b=c\wedge(a\vee b) \quad \text{for every} \quad c\geqq b).$$

(Note that $(a, b)M^*$ is equivalent to $(b, a)M^*$ in the sense of [4].)

A lattice $L$ is called *M-symmetric* (resp. *M*-symmetric*) when $(a, b)M$ implies $(b, a)M$ (resp. $(a, b)M^*$ implies $(b, a)M^*$) in $L$. $L$ is called *cross-symmetric* (resp. *dual cross-symmetric*) when $(a, b)M$ implies $(b, a)M^*$ (resp. $(a, b)M^*$ implies $(b, a)M$) in $L$.

**Lemma 1.** *Let $a$, $b$ and $c$ be elements of a lattice $L$.*

(i) *If $(a, b)M$ and $(a\wedge b, c)M$ then $(a_1, b\wedge c)M$ for any element $a_1$ of the interval $L[a\wedge c, a]$.*

(ii) *If $(a, b)M$ then $(a_1, b_1)M$ for any $a_1\in L[a\wedge b, a]$ and $b_1 \in L[a\wedge b, b]$.*

**Proof.** (i) Let $a\wedge c\leqq a_1\leqq a$. Then $a_1\wedge c=a\wedge c$. If $d\leqq b\wedge c$, then by $(a, b)M$ and $(a\wedge b, c)M$ we have

$$\begin{aligned}(d\vee a_1)\wedge(b\wedge c)&\leqq(d\vee a)\wedge b\wedge c=\{d\vee(a\wedge b)\}\wedge c\\&=d\vee(a\wedge b\wedge c)=d\vee(a_1\wedge b\wedge c)\leqq(d\vee a_1)\wedge(b\wedge c).\end{aligned}$$

Hence $(a_1, b\wedge c)M$.

(ii) Assume $(a, b)M$ and let $a\wedge b\leqq b_1\leqq b$. Since $(a\wedge b, b_1)M$, it follows from (i) that

$$(a_1, b_1)M \quad \text{for any} \quad a_1\in L[a\wedge b_1, a]=L[a\wedge b, a].$$