49. *Semi-groups of Nonlinear Operators on Closed Convex Sets*

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(Comm. by Zyoiti SUETUNA, M. J. A., April 12, 1969)

Let H be a real or complex Hilbert space, whose inner product and norm are denoted by $\langle\ ,\ \rangle$ and $|\ \ |$. Let $D\subseteq H$ and let $\{T_t;\, t\geq 0\}$ be a semi-group of mappings of D into itself. For $K\subseteq H$ and $x\in H$, let $C_K(x)$ be the strong closure of the set $\{r(z-x);\, r\geq 0,\, z\in K\}$. If K is a closed subset of D and if $T_tK\subseteq K$ for $t\geq 0$, it is obvious that we have

$$A_0x = s-\lim_{h\downarrow 0}\frac{1}{h}(T_hx - x)\in C_K(x),\ x\in\partial K \tag{1}$$

whenever the strong limit exists. We are interested in the converse problem: under what conditions on $\{T_t\}$ and K does (1) imply $T_tK\subseteq K$, $t\geq 0$. We consider this converse problem when $\{T_t\}$ is a strongly continuous semi-group of contractions and K is a closed convex set.[1] Our result is given in Theorem 1 below. Theorem 1 enables us to prove Theorem 2 which gives a sufficient condition on a (nonlinear) dissipative operator that it be a strong generator of a semi-group of contractions on a closed convex set. Finally we apply Theorem 2 to prove Theorem 3 which shows the existence of a sequence of semi-groups with continuous infinitesimal generators approximating a given semi-group of contractions on a closed convex set.

We begin with a simple lemma which we shall need later.

Lemma 1. *Let $u\in H$ and K be a closed convex subset of H. Then there exists one and only one element $v\in K$ such that $|u-v| = \inf\{|u-z|;\, z\in K\}$. Moreover, we have the inequality*

$$\mathrm{Re}\,\langle y, u-v\rangle\leq 0,\ y\in C_K(v). \tag{2}$$

Proof. We shall prove the inequality (2). For $y\in C_K(v)$ we can find a sequence $r_n\geq 0$ and a sequence $z_n\in K$ such that $y=s-\lim_{n\to\infty} r_n(z_n-v)$. Since $v+c(z_n-v)\in K$ for $0\leq c\leq 1$, it follows that

$$|u-v-c(z_n-v)|\geq |u-v|,\qquad 0\leq c\leq 1.$$

This gives

$$\mathrm{Re}\,\langle z_n-v, u-v\rangle\leq 0.$$

1) It is to be noted that (1) does not necessarily imply $T_tK\subseteq K$, $t\geq 0$, if we do not assume that T_t is a contraction for $t\geq 0$.