84. *Continuity in Mixed Norms*

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(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1969)

1. Consider a normed vector lattice X over the real number field R with $X\supset R$. The norm $\|\circ\|_X$ on X is supposed to satisfy

(1) $$\min(\alpha, \|x\|_X)\leq\|(\alpha\vee x)\wedge\beta\|_X \leq\min(\beta, \|x\|_X)\ (\forall x\in X, \forall\alpha, \forall\beta\in R, 0\leq\alpha\leq\beta).$$

Take a seminorm p_X on X satisfying the following:

(2) $$p_X(\alpha)=0\ (\forall\alpha\in R);$$

(3) $$p_X^2(x)=p_X^2(x\wedge\alpha)+p_X^2(x\vee\alpha)\ (\forall x\in X, \forall\alpha\in R);$$

(4) $$\lim_{\alpha'\uparrow\alpha,\beta'\downarrow\beta}p_X((\alpha'\vee x)\wedge\beta')=p_X((\alpha\vee x)\wedge\beta)\ (\forall x\in X, \forall\alpha, \forall\beta\in R).$$

With the aid of p_X we can define a new norm in X:

(5) $$|||x|||_X=\|x\|_X+p_X(x).$$

Let Y, $\|\circ\|_Y$, p_Y, and $|||\circ|||_Y$ be as above. Then we can show the following

Theorem. *Suppose that T is an isomorphism of $(X, \|\circ\|_X)$ onto $(Y, \|\circ\|_Y)$ as normed vector lattices with $T(\alpha)=\alpha(\forall\alpha\in R)$. Then*

(6) $$\exists K: K^{-1}p_X(x)\leq p_Y(T(x))\leq Kp_X(x)\ (\forall x\in X)$$

if and only if

(7) $$\exists K: K^{-1}|||x|||_X\leq|||T(x)|||_Y\leq K|||x|||_X\ (\forall x\in X).$$

Proof. Since $\|T(x)\|_Y=\|x\|_X$, (6) clearly implies (7). To show the reversed implication let $A=\{x\in X\ x\geq0, \|x\|_X\leq1\}$. Then from (7) it follows that

(8) $$\exists K: p_Y(T(x))\leq K(1+p_X(x))\ (\forall x\in A).$$

Fix an arbitrary $x\in A$ and define

$$x_i=n\left(\left(\frac{i-1}{n}\vee x\right)\wedge\frac{i}{n}-\frac{i-1}{n}\right)\ (i=1, 2, \cdots, n).$$

By (1), $x_i\in A$ $(i=1, 2, \cdots, n)$. Since T is an isomorphism of vector lattices with $T(\alpha)=\alpha(\alpha\in R)$,

$$T(x_i)=n\left(\left(\frac{i-1}{n}\vee T(x)\right)\wedge\frac{i}{n}-\frac{i-1}{n}\right)(i=1, 2, \cdots, n).$$

In view of (2), we see that

$$p_X(x_i)=np_X\left(\left(\frac{i-1}{n}\vee x\right)\wedge\frac{i}{n}\right),\quad p_Y(T(x_i))=np_Y\left(\left(\frac{i-1}{n}\vee T(x)\right)\wedge\frac{i}{n}\right).$$

Repeated use of (3) yields

$$p_X^2(x)=\sum_{i=1}^n p_X^2\left(\left(\frac{i-1}{n}\vee x\right)\wedge\frac{i}{n}\right),\quad p_Y^2(T(x))=\sum_{i=1}^n p_Y^2\left(\left(\frac{i-1}{n}\vee T(x)\right)\wedge\frac{i}{n}\right)$$