81. On Generalized Integrals. V

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Many real variable integrals have been defined as extensions of the Lebesgue integral. Some of them depend heavily on special properties of the derivative on the real line, and some follow the idea of definite integral as limits of certain approximating sums. The idea of Kunugi's generalized integrals follows the latter direction. As it is already known (IV,[1] Theorem 7) [4], for the $(E.R.\varphi)$ integrable function $f(x)$, the integral is defined as limit of approximating sums $\int_{G_n} f(x)dx$, $G_n=\{x;\, f(x)\leq n\varphi(x)\}$. In this case, the sequence $\{G_n\}$ satisfies the condition $\lim_{n\to\infty} n\int_{G_n}\varphi(x)dx=0$. Theorem 9 shows that the Denjoy-integrable function $f(x)$ in the general (resp. special) sense in $[a, b]$ is a measurable function for which there exists a monotone increasing sequence $\{F_n\}$, with union $[a, b]$, of closed sets with the properties [C] and [D] (resp. $[D_*]$), and then the integral is given as limit of approximating sums $\int_{F_n} f(x)dx$. On the other hand, we have seen that for each positive Lebesgue-integrable function φ, the sets of $(E.R.\varphi)$ integrable functions and Denjoy-integrable functions in either the general or the special sense partially intersect. Moreover, for some functions which are both $(E.R.\varphi)$ and Denjoy-integrable, even where the Denjoy integrability is in the special sense, both integrals do not coincide.[2] Thus, we see that the $(E.R.)$ integrability differs essentially from the Denjoy integrability, and the difference between the methods of totalization of the $(E.R.)$ integral and the Denjoy

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1) The reference number indicates the number of the Note.

2) This has been proved by I. A. Vinogradova for the case of special $(E.R.)$ integral (i. e. A-integral) (see [6]). On the other hand, if $\varphi(t)$ is a positive Lebesgue-integrable function in $[a, b]$, and if $\Phi(t)$ is the indefinite integral of $\varphi(t)$ such that $\Phi(a)=\alpha$ and $\Phi(b)=\beta$, then for a function $f(x)$ defined in $[\alpha, \beta]$, we see that: (1) $f(\Phi(t))\varphi(t)$ is $(E.R.\varphi)$ integrable in $[a, b]$ if and only if $f(x)$ is $(E.R.)$ integrable in the special sense in $[\alpha, \beta]$, and then $(E.R.\varphi)\int_a^b f(\Phi(t))\varphi(t)dt=(E.R.)\int_\alpha^\beta f(x)dx$ (see IV), and (2) $f(\Phi(t))\varphi(t)$ is Denjoy-integrable in the general (resp. special) sense in $[a,b]$ if and only if $f(x)$ is Denjoy-integrable in the general (resp. special) sense in $[\alpha, \beta]$, and then $(D)\int_a^b f(\Phi(t))\varphi(t)dt=(D)\int_\alpha^\beta f(x)dx$. Therefore, it follows that the assertion is also true for the general case.