74. *Boundedness of Solutions to Nonlinear Equations in Hilbert Space*

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In what follows, by $H=(H, \langle , \rangle)$ we denote a complex Hilbert space, and by $B=B(H, H)$, the space of all bounded linear operators from H into H, associated with the strong operator topology. The only topology that we consider on H is the strong one.

Our aim in this paper is to give a boundedness theorem for the solutions of the differential equation

$$(*) \qquad \dot{x}=A(t)x+f(t, x),$$

where $x: I\to H$, $I=[t_0, +\infty)$, $t_0\geq 0$, is a differentiable function on I with continuous first derivative,[2)] $A: I\to B$ is a continuous function on I, and $f: I\times H\to H$ is also continuous on $I\times H$.

1. **Theorem 1.** *Consider* $(*)$ *under the following assumptions*:

(i) *there exists an operator valued function* $Q: I\to B$ *continuous and such that*:

$$(\mathrm{i}_1) \qquad \dot{Q}(t)+Q(t)A(t)+A^*(t)Q(t)=0,^{3)} \qquad t\in I,$$

and

$$(\mathrm{i}_2) \qquad |\langle Q(t)x, x\rangle|\geq g(\|x\|), \qquad (t, x)\in I\times H,$$

where $g: \boldsymbol{R}_+\to\boldsymbol{R}_+=[0, +\infty)$ *is continuous and* $\limsup_{y\to+\infty} g(y)=+\infty$;

(ii) $\|x\|\cdot\|f(t, x)\|\leq p(t)g(\|x\|)$, *with* $p: I\to\boldsymbol{R}_+$ *continuous and such that*

$$\int_{t_0}^{\infty} p(t)\|Q(t)\|dt<+\infty;$$

then, if $x(t)$, $t\in I$, *is a solution of* $(*)$, *it is bounded, i.e. there exists a constant* $k>0$ *such that* $\|x(t)\|\leq k$ *for every* $t\in I$.

Proof. By differentiation of the function

$$(1) \qquad V(t)=\langle Q(t)x(t), x(t)\rangle,$$

we have

$$\begin{aligned}
\dot{V}(t)&=\langle \dot{Q}(t)x(t)+Q(t)\dot{x}(t), x(t)\rangle+\langle Q(t)x(t), \dot{x}(t)\rangle\\
&=\langle \dot{Q}(t)x(t)+Q(t)A(t)x(t)+Q(t)f(t, x(t)), x(x)\rangle\\
(2) \qquad &\quad+\langle Q(t)x(t), A(t)x(t)+f(t, x(t))\rangle\\
&=\langle(\dot{Q}(t)+Q(t)A(t)+A^*(t)Q(t))x(t), x(t)\rangle\\
&\quad+\langle Q(t)f(t, x(t)), x(t)\rangle+\langle Q(t)x(t), f(t, x(t))\rangle
\end{aligned}$$

and by integration from t_0 to t $(t_0\leq t)$, we have

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2) The existence of solutions on I is assumed without further mention.

3) $A^*(t)$ is the adjoint of the operator $A(t)$.