97. A Remark on the Π-imbedding of Homotopy Spheres

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Let Θ_n be the group of homotopy n-spheres and $\tilde{S}^n$ be an element of Θ_n. $\tilde{S}^n$ represents an element of a subgroup $\Theta_n(\partial\pi)$ of Θ_n if and only if $\tilde{S}^n$ is the boundary of a parallelizable manifold.

It is known that every $\tilde{S}^{13}$ is imbeddable in the 17-dimensional unit sphere S^{17} with a trivial normal bundle (Katase [3]). (Such an imbedding is called a π-imbedding.) But in the case of codimension 3, it has been unknown whether the π-imbedding exists or not. The result of this paper is that there exists a 13-dimensional homotopy sphere $\tilde{S}^{13}$ which is not π-imbeddable in S^{16}.

1. Suppose that $\tilde{S}^n$ is π-imbedded in S^{n+k} $(3 \leq k < n)$. Then the tubular neighbourhood of $\tilde{S}^n$ in S^{n+k} and its boundary is easily seen to be diffeomorphic to $S^n \times D^k$ and $S^n \times S^{k-1}$ respectively (here D^k is the closed unit disk in euclidean k-space and is bounded by S^{k-1}). Moreover, $\tilde{S}^n$ is isotopic to an $\tilde{S}_1^n$ which lies in $S^n \times S^{k-1} \subset S^{n+k}$ with normal $(k-1)$-frame $\mathcal{F}$ in $S^n \times S^{k-1}$ and is homotopic, in $S^n \times S^{k-1}$, to $S^n \times x_0$ for some $x_0 \in S^{k-1}$ (Levine [6]). The Pontrjagin-Thom construction with respect to a normal $(k-1)$-frame $\mathcal{F}$ on $\tilde{S}_1^n$ in $S^n \times S^{k-1}$ yields a map

$$\varphi\,;\, S^n \times S^{k-1} \longrightarrow S^{k-1}$$

which maps $\tilde{S}_1^n$ to a point p in S^{k-1} (see, for example, Kervaire [4]).

Suppose that φ can be extended to a map

$$\Phi'\,;\, S^{n+k} - \mathrm{Int}\, S^n \times D^k \longrightarrow S^{k-1}.$$

Then we can approximate it by a smooth map Φ keeping φ fixed.

Since we may consider p as a regular value of Φ, $\Phi^{-1}(p)$ or at least the component of $\tilde{S}_1^n$ in $\Phi^{-1}(p)$ is an $(n+1)$-dimensional submanifold of S^{n+k} with a trivial normal bundle and its boundary is $\tilde{S}_1^n$. Therefore $\tilde{S}^n$ bounds a parallelizable manifold, i.e., $\tilde{S}^n$ is an element of $\Theta_n(\partial\pi)$.

2. Now we consider the obstructions to extending φ over $S^{n+k} - \mathrm{Int}(S^n \times D^k)$ which lie in the cohomology groups

$$H^r(S^{n+k} - \mathrm{Int}(S^n \times D^k),\, S^n \times S^{k-1}\,;\, \pi_{r-1}(S^{k-1})).$$