94. *On Dirichlet Spaces and Dirichlet Rings*

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In [2], we have introduced the notion of the Dirichlet space relative to an L^2-space (we will call this an L^2-Dirichlet space). The purpose of this paper is to derive a normed ring (called a Dirichlet ring) from any given L^2-Dirichlet space in the similar manner as Royden ring [5] from the space of functions with finite Dirichlet integrals. Dirichlet rings will enable us to define a natural equivalence relation among the collection of all L^2-Dirichlet spaces. We will discuss elsewhere the problem to find out nice versions from each equivalence class ([3]).

§ 1. L^2-Dirichlet spaces and L^2-resolvents.

We call $(X, m, \mathcal{F}, \mathcal{E})$ a complex L^2-*Dirichlet space* (in short, a *D-space*) if the following conditions are satisfied.

(1.1) X is a locally compact Hausdorff space.

(1.2) m is a Radon measure on X.

(1.3) $\mathcal{F}$ is a linear subspace of complex $L^2(X)=L^2(X; m)$, two functions being identified if they coincide m-a.e. on X. $\mathcal{E}$ is a non-negative definite bilinear form on $\mathcal{F}$ and, for each $\alpha>0$, $\mathcal{F}$ is a complex Hilbert space with inner product

$$\mathcal{E}^\alpha(u, v)=\mathcal{E}(u, v)+\alpha(u, v)_X,$$

where $(u, v)_X$ is the inner product in $L^2(X)$-sense.

(1.4) Each normal contraction operates on $(\mathcal{F}, \mathcal{E})$: if $u \in \mathcal{F}$ and a measurable function v satisfies

$$|v(x)| \leqq |u(x)|, \quad |v(x)-v(y)| \leqq |u(x)-u(y)| \ m\text{-a.e},$$

then $v \in \mathcal{F}$ and $\mathcal{E}(v, v) \leqq \mathcal{E}(u, u)$.

Let (X, m) be as above. We call a family of linear bounded symmetric operators $\{G_\alpha, \alpha>0\}$ on $L^2(X)$ an L^2-*resolvent* iff it satisfies the resolvent equation and it is sub-Markov: for each $\alpha>0$, G_α translates each real function into a real function and $0 \leqq \alpha G_\alpha u \leqq 1$ m-a.e for $u \in L^2(X)$ such that $0 \leqq u \leqq 1$ m-a.e.

There is a one-to-one correspondence between the class of D-spaces and the class of L^2-resolvents ([2]).

In fact, with any D-space $(X, m, \mathcal{F}, \mathcal{E})$, we can associate an L^2-resolvent by the equation

(1.5) $$\mathcal{E}^\alpha(G_\alpha u, v)=(u, v)_X \quad \text{for any} \quad v \in \mathcal{F},$$