126. Some Theorems on Certain Contraction Operators

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1. Let H be a complex Hilbert space. An operator T means a bounded linear operator on H. In this paper we shall prove some theorems on certain contraction operators and related results in consequence of these theorems.

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2. In this section, at first we shall begin to define the classes of operators as follows.

Definition 1. *An operator T is said to be normaloid in the sense that T satisfies*

(1) $$\|T^n\| = \|T\|^n \qquad n=1, 2, \cdots$$

equivalently, the spectral radius $r(T)$ is equal to $\|T\|$ ([3]).

Definition 2. *An operator T is said to be paranormal in the sense that T satisfies*

(2) $$\|T^2x\| \geqq \|Tx\|^2 \qquad \textit{for every unit vector } x \textit{ in } H.$$

In [4] this operator is named an operator of class (N).

It is known that this class of paranormal operators properly includes that of hyponormal operators and is properly included in the class of normaloids [1]-[3].

We shall discuss the following theorem and its consequence.

Theorem 1. *An idempotent normaloid operator T is a projection.*

To prove Theorem 1, we need the following already known theorem ([8]).

Theorem 2. *If T is an idempotent and contraction operator ($\|T\| \leqq 1$), then T is a projection.*

The following proof of Theorem 2, based on a method of Mlak [5], which is originally due to von Neumann, is simpler than that appeared in the literature.

Proof of Theorem 2.

$$\begin{aligned}\|Tx - T^*Tx\|^2 &= \|Tx\|^2 - (Tx, T^*Tx) - (T^*Tx, Tx) + \|T^*Tx\|^2 \\ &= \|Tx\|^2 - (T^2x, Tx) - (Tx, T^2x) + \|T^*Tx\|^2 \\ &= \|Tx\|^2 - \|Tx\|^2 - \|Tx\|^2 + \|T^*Tx\|^2 \\ &= \|T^*Tx\|^2 - \|Tx\|^2 \leqq 0\end{aligned}$$

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