125. A Note on Two Inequalities Correlated to Unitary ρ-Dilatations

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1. In [7] Sz-Nagy and C. Foias introduced the notion of the class $\mathcal{C}_\rho$ as follows. For each fixed $\rho>0$, $\mathcal{C}_\rho$ is the class of operators T on a given complex Hilbert space H having the following property:

There exist a Hilbert space K containing H as a subspace and a unitary operator U on K satisfying the following representation

$$(*)\qquad T^n=\rho PU^n \ (n=1,2,\cdots)$$

where P is the orthogonal projection of K on H.

It is well known that $\mathcal{C}_1=\{T: \|T\|\leqq 1\}$ ([6]) and $\mathcal{C}_2=\{T: \|T\|_N\leqq 1\}$ ([1]) where $\|T\|_N$ means the numerical radius of T,

$$\|T\|_N=\sup|(Th,h)| \quad \text{for every unit vector } h \text{ in } H.$$

The following theorem is known and we cite for the sake of convenience ([5] [7]).

Theorem A. (i) *For each fixed $\rho>0$ and T on $\mathcal{L}(H)$, $T\in\mathcal{C}_\rho$ if and only if*

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}T_\rho(n)\geqq 0 \quad \textit{for every } \theta \textit{ and } r \textit{ such that } 0\leqq r<1.$$

$$\textit{where}\quad T_\rho(n)=\begin{cases}\dfrac{1}{\rho}T^n & (n\geqq 1)\\ I & (n=0)\\ \dfrac{1}{\rho}T^{*-n} & (n\leqq -1)\end{cases}$$

(ii) *$\mathcal{C}_\rho$ is non-decreasing with respect to the index ρ in the sense that $\mathcal{C}_{\rho_1}\subset\mathcal{C}_{\rho_2}$ if $0<\rho_1<\rho_2$.*

In [5] J. Holbrook has defined the function w_ρ as follows

$$w_\rho(T)=\inf\left\{u;\, u>0 \quad \frac{1}{u}T\in\mathcal{C}_\rho\right\}.$$

Concerning to this function $w_\rho(T)$ he has proved the following theorems

Theorem B. *$w_\rho(T)$ has the following properties:*

(1) $w_\rho(T)<\infty$

(2) $w_\rho(T)>0$ *unless* $T=0$; *in fact* $w_\rho(T)\geqq\dfrac{1}{\rho}\|T\|$

(3) $w_\rho(zT)=|z|\,w_\rho(T)$

(4) $w_\rho(T)\leqq 1$ *if and only if* $T\in\mathcal{C}_\rho$

(5) $w_\rho(T)$ *is a norm whenever* $0<\rho\leqq 2$