124. *On the Critical Points of Harmonic Functions*

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1. We admit as 2-cells the homeomorph of any convex polygon,[1] regarding the vertices and edges of this image as 0 and 1-cells respectively. An i-complex is a connected set of a finite number of i-cells ($i=1, 2$) and the characteristic ρ of a complex is defined as $\rho=-a_0+a_1-a_2$ where a_i is the number of i-cells ($i=0, 1, 2$) in the complex. The object of this paper is to give another proof to a theorem of Nevanlinna[2] on harmonic functions and to show that the characteristic of a domain plays an important rolle.

Let D be a domain or the union of a finite number of domains and $\bar{D}$ be its closure. We divide $\bar{D}$ in a finite number of 2-cells and consider $\bar{D}$ as a union of 2-complexes. We denote by a_i and a'_i the number of i-cells ($i=0, 1, 2$) contained in $\bar{D}$ and D respectively. Then $\rho(\bar{D})=-a_0+a_1-a_2$ and $\rho(D)=-a'_0+a'_1-a'_2$ are the sums of the characteristics of all connected components of $\bar{D}$ and D respectively. A 1-complex representing a simple closed curve has the same number of 0-cells as 1-cells and so contributes nothing to the characteristic. Hence we have $\rho(D)=\rho(\bar{D})$, when the boundary of D consists of a finite number of simple closed curves.

Let $u(z)$ be a harmonic function in a domain D and $C(u)$ be the niveau curve: $u(z)=\text{const.}=u$. The critical points of $u(z)$ in the ordinary sense are the points $z=x+iy$ at which $\dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial y}=0$. Let $v(z)$ be the conjugate harmonic function of $u(z)$ and $w(z)=u(z)+iv(z)$. Then, by virtue of Cauchy-Riemann differential equation, such a point is a zero of $w'(z)$. The order of the zero is said to be the multiplicity of a critical point. Let $k-1$ be the multiplicity of a critical point z_0 of $u(z)$, then the niveau curve $C(u)$ through z_0 consists of k curves neighbouring z_0, each making an angle of $\dfrac{\pi}{k}$ at z_0 with its successor.

2. Let D be a domain bounded by m simple closed curves $C_1, C_2, \cdots, C_m$ and α be a set of a finite number of arcs on the boundary of D. We denote by n the number of arcs contained in α, which do not coinside with any of the whole curve C_i. Let $u(z)=\omega(z, \alpha, D)$ be the harmonic measure of α at the point z in D. We have $0<u(z)<1$ in D