122. Korteweg-deVries Equation. I

Global Existence of Smooth Solutions

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In this note we state the result only. The detailed proof will be published elsewhere. We treat in this note (I) the global existence of smooth solutions of the Cauchy problem for the KdV equation. In the following note (II) [6] we show the existence of computable approximate solutions by the method of finite difference schemes.

Consider the Cauchy problem for the KdV equation.

$$(1) \qquad D_t u = uDu + D^3u + g(x, t) \qquad (x, t) \in R^1 \times (0, \infty)$$

$$(2) \qquad u(x, 0) = f(x) \qquad x \in R^1$$

Here $D_t = \dfrac{\partial}{\partial t}$, $D = \dfrac{\partial}{\partial x}$

Main theorem.

If

$$f(x) \in \mathcal{E}_{L^2}^{3(k+1)}(R^1)$$

$$g(x, t) \in \mathcal{E}_t^{k+1}(L^2) \cap [\mathcal{E}_t^k(L^2) \cap \mathcal{E}_t^{k-1}(\mathcal{E}_{L^2}^3) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k})]$$

then there exists uniquely the solution $u(x, t)$ of the Cauchy problem for the KdV equation for $0 \leq t < \infty$ such that

$$u(x, t) \in \mathcal{E}_t^k(L^2) \cap \mathcal{E}_t^{j-1}(\mathcal{E}_{L^2}^3) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3k}).$$

Using Sobolev's lemma we conclude easily following corollaries

Corollary 1.

If

$$f(x) \in \mathcal{E}_{L^2}^{3(k+3)}(R^1)$$

$$g(x, t) \in \mathcal{E}_t^{k+3}(L^2) \cap [\mathcal{E}_t^{k+2}(L^2) \cap \cdots \cap \mathcal{E}_t^0(\mathcal{E}_{L^2}^{3(k+2)})]$$

then for any i, j such that $i + j \leq k$

$$D_t^i D^{3j} u \in \mathcal{B}^0(R^1 \times [0, T]) \quad \textit{for } {}^\forall T > 0.$$

Remark. In Corollary 1 if we take $k=1$ we obtain the global existence theorem of classical solutions of KdV equation.

Corollary 2.

If

$$f(x) \in \mathcal{E}_{L^2}^\infty$$

$$g(x, t) \in \mathcal{E}_t^\infty(\mathcal{E}_{L^2}^\infty)$$

then

$$u(x, t) \in \mathcal{E}_t^\infty(\mathcal{E}_{L^2}^\infty)$$

especially

$$u(x, t) \in \mathcal{B}^\infty(R^1 \times [0, T]) \quad \textit{for } {}^\forall T > 0.$$