118. *On a General Form of the Weyl Criterion in the Theory of Asymptotic Distribution. II*

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V. **Applications.** **1.** Let u_n $(n=1, 2, \cdots)$ be a sequence of real numbers. Then define the function f on $[0, \infty)$ as follows:

$$f(n)=u_n \qquad (n=1, 2, \cdots),$$
$$f(t)=f([t]+1) \qquad (t\not\equiv 0 \pmod 1).$$

Let $B(t)=[t]$ $(t\not\equiv 0 \pmod 1)$ and continuous on the left for every t. Then using the same notation as in IV we have

$$F_T(\xi)=\frac{1}{B(T)}\int_0^T \chi_{[0,\xi)}((f(t)))dB(t)$$

(where the integral is taken over the interval $[0, T)$)

$$=\frac{1}{[T]}\sum_{\substack{n=1\\ 0\le f(n)<\xi}}^{[T]} 1$$

Let the d.f. $F(\xi)$ equal to $\xi(0\le\xi\le 1)$, and $=0$ $(\xi\le 0)$ and $=1$ $(\xi\ge 1)$. Then $F_T(\xi)\xrightarrow{c}\xi$, as $T\to\infty$, if and only if for $k=1, 2, \cdots$

$$\lim_{T\to\infty}\frac{1}{[T]}\int_0^T \exp 2\pi ikf(t)d[T]$$
$$=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\exp 2\pi iku_n=\int_0^1 \exp 2\pi ikxdx=0,$$

or, Theorem 5 implies the Weyl criterion for the uniform distribution mod 1 of a sequence of real numbers. See [1].

2. Let u_n, $f(t)$, $B(t)$ and $F_T(\xi)$ be defined as in **1**. Let $F(\xi)$ be a d.f. with $F(\xi)=0$ $(0\le\xi<1)$ and $F(\xi)=1$ $(\xi>1)$. Suppose furthermore that $\Delta F(0)=\Delta F(1)=0$. Then

$$F_T(\xi)\xrightarrow{c}F(\xi), \qquad \text{as } T\to\infty,$$

if and only if

$$\lim_{T\to\infty}\frac{1}{B(T)}\int_0^T \exp 2\pi ikf(t)dB(t)=\lim_{T\to\infty}\frac{1}{[T]}\sum_{n=1}^{[T]}\exp 2\pi iku_n$$
$$=\int_0^1 \exp 2\pi ikxdF(x).$$

Moreover

$$F_T(\xi)=\frac{1}{B(T)}\int_0^T \chi_{[0,\xi)}((f(t)))dB(t)=\frac{1}{[T]}\sum_{\substack{n=1\\ 0\le f(n)<\xi}}^{[T]} 1.$$

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