117. *On a General Form of the Weyl Criterion in the Theory of Asymptotic Distribution.* I

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I. Introduction. There are a few examples of using probability theory methods in the theory of asymptotic distribution mod 1, and it seems quite natural to do so, since the theory in question deals with the problem of the distribution of the fractional parts of the elements of a sequence of real numbers and of the values of a realvalued function defined on $[0, \infty)$. In fact, the classical Weyl criterion concerning sequences of real numbers ([1]) has been viewed from the point of probability theory ([2]-[4]) and in [5] the method used in [4] is applied to the case of distributing the values mod 1 of functions defined on $[0, \infty)$. In the present paper we apply (Theorem 5) a (classic) result on the convergence of a class of distribution functions in order to find a very general form of the Weyl criterion, covering even the case of the Niven-Uchiyama criterion ([6], [7]) in the theory of the distribution of sequences of integers modulo m, and also the case of some summability-procedure distribution of sequences of real numbers, such as the Borel summability distribution of a sequence of real numbers (Application 6).

II. Definitions. Let L denote either the interval $[0, \infty)$ or the sequence $1, 2, \dots$.

The function $F(x)$ defined on the extended real line is said to be a *distribution function* (abbreviated d.f.) if $F(x)$ is bounded, non-decreasing and continuous on the left. We have then $F(-\infty)=\lim_{x\to-\infty} F(x)$ and $F(\infty)=\lim_{x\to\infty} F(x)$.

Let for each $t \in L$ a d.f. $F_t(x)$ be defined.

Now $F_t(x)$ is said to *converge weakly* to a d.f. $F(x)$ as $t\to\infty$, or $F_t(x)\xrightarrow{w}F(x)$, if

$$\lim_{t\to\infty} \{F_t(b)-F_t(a)\}=F(b)-F(a) \tag{1}$$

for every pair of continuity points a and b of $F(x)$. (See [8], p. 76, where the notion of vague convergence is defined, slightly different from the above notion of weak convergence. Also slightly different is Loève's description of weak convergence ([3], p. 178), where

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