114. A Note on Variation Theory

Dedicated to Professor Atuo Komatu on his 60th birthday

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This note is a contribution to the study of variation problem and is intended to give a new version of the classical variation theory, due essentially to Lusternik-Schnirelmann.

We assume throughout this note that a space X is Hausdorff and locally connceted in all dimensions, that is, for every point $x \in X$ there is a neighbourhood U of x such that $H_*(U, Z)=H_*(x, Z)$, where $H_*(\ , Z)$ stands for integral homology group.

Let φ be a non-negative continuous function on X and let f be a map of X into itself. The triple $(X, \varphi; f)$ is said to be a variation problem of type C (over a field k), if X, φ and f satisfy the following conditions:

A) $\varphi(f(x)) \leqq \varphi(x)$ for any $x \in X$.

B) $\varphi(f(x))=\varphi(x)$ implies $\varphi(f \circ f(x))=\varphi(x)$.

C) $f_*=\mathrm{id}: H_*(X, k) \to H_*(X, k)$.

Given a variation problem $(X, \varphi; f)$, we define for a compact set $A \subset X$ as follows:

$$\varphi(A)=\sup_{a \in A} \varphi(a), \qquad |A|=\inf_n \varphi(f^n(A))$$

$$\gamma=\{x \in X \mid \varphi(f(x))=\varphi(x)\}$$

$$F(A)=\overline{\bigcup_n f^n(A)}.$$

A point x in γ is called a φ-stationary point.

For $Y \subset X$, let

$$Y(a)=\{y \in Y \mid \varphi(y)=a\},$$

$$Y([a, b])=\{y \in Y \mid a \leqq \varphi(y) \leqq b\},$$

$$Y((-\infty, b])=\{y \in Y \mid \varphi(y) \leqq b\}.$$

Existence lemma. *Let $(X, \varphi; f)$ be a variation problem of type C and let A be a compact subset of X such that $F(A) \cap X([|A|, \infty))$ is compact.*

Then

$$\gamma(|A|) \cap F(A) \neq \phi.$$

Proof. Take $a_n \in A$ so that

$$\varphi(f^n(a_n))=\varphi(f^n(A)) \geqq |A|$$

and set $y_n=f^n(a_{n+1})$. Then the inequalities

$$\varphi(y_n) \geqq \varphi(f(y_n))=\varphi(f^{n+1}(a_{n+1})) \geqq |A|$$