152. On Tensor Products of Operators

By Takayuki FURUTA and Ritsuo NAKAMOTO
Faculty of Engineering, Ibaraki University, Hitachi
and Tennoji Senior High School, Osaka

(Comm. by Kinjirô KUNUGI, M. J. A., Oct. 13, 1969)

1. Introduction. In this paper we shall discuss the tensor products of bounded linear operators on a complex Hilbert space H.

Following after Halmos [2], we define the numerical radius $\|T\|_N$ and the numerical range $W(T)$ as follows:

$$\|T\|_N = \sup |W(T)|$$
$$W(T) = \{(Tx, x) ; \|x\| = 1\}.$$

Definition 1. An operator T is said to be *normaloid* if

$$\|T\| = r(T),$$

where $r(T)$ means the spectral radius of T, or equivalently

$$\|T^n\| = \|T\|^n \ (n = 1, 2, \cdots\cdots).$$

Definition 2. An operator T is said to be *spectraloid* if

$$\|T\|_N = r(T),$$

or equivalently

$$\|T^n\|_N = \|T\|_N^n \ (n = 1, 2, \cdots\cdots) \ ([4]).$$

Definition 3. An operator T is said to be *convexoid* if

$$\overline{W(T)} = \mathrm{co}\, \sigma(T),$$

where the bar denotes the closure and co $\sigma(T)$ means the convex hull of the spectrum $\sigma(T)$ of (T).

It is known that the classes of normaloids and convexoids are both contained in the class of spectraloids ([2]).

In recent years several authors paid attention to the spectral properties of the tensor products of operators on H; Brown and Pearcy [1] established

Theorem A. *If $\sigma(T)$ and $\sigma(S)$ are spectra of operators T and S respectively, then*

$$\sigma(T \otimes S) = \sigma(T) \cdot \sigma(S).$$

In connection with Theorem A, T. Saitô also proved analogous theorems among the numerical ranges of T, S and $T \otimes S$ as follows.

Theorem B ([5]).

(i) *For arbitrary operators T and S on a Hilbert space H, then*

$$\overline{W(T \otimes S)} \supseteq \overline{\mathrm{co}}\, (W(T) \cdot W(S))$$

where $\overline{\mathrm{co}}\, Z$ means the closure of convex hull of the set Z.

(ii) *Let T and S be operators on a Hilbert space H, then the condition that $T \otimes S$ is convexoid implies*