143. On a Property of $p-1$

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Erdös [1] proved in an ingenious manner that the density of the integers having a divisor between n and $2n$ tends to zero as n tends to infinity.

The purpose of this short note is to prove that the same fact holds for the sequence $\{p-1\}$, where p denotes a prime. More precisely we shall prove the following

Theorem. *The density, with respect to the sequence of all primes, of the prime p such that $p-1$ has a divisor between n and $n \exp(h^{-1}(n) \operatorname{loglog} n)$ tends to zero as n tends to infinity, where $h(n)$ is an arbitrary increasing function such that $h(n)\to\infty$ and $h^{-1}(n) \operatorname{loglog} n\to\infty$ as $n\to\infty$.*

For the proof of the theorem we need three lemmas:

Lemma 1. *Let $\omega(m)$ be the number of all prime divisors of m. Then, if $1/2\leq a<1$, we have*

$$\sum_{\substack{n\leq m\leq n \exp(h^{-1}(n)\operatorname{loglog} n)\\ \omega(m)\leq \alpha \log\log n}} m^{-1}=0\{\log^{\sigma_\alpha -1} n \log\log n\},$$

where $\gamma_a=a-a\log a$.

This is a trivial modification of Lemma 7 of Hooley [2].

Lemma 2. *Let $\omega_n(m)$ be the number of all prime divisors less than n of m. Then for $n\leq \log x$ we have*

$$\sum_{p\leq x} (\omega_n(p-1)-\log\log n)^2=0(\pi(x)\log\log n),$$

where $\pi(x)$ is the number of primes not exceeding x.

Lemma 3. *If c and n are less than $\log x$, then we have*

$$\sum_{\substack{p\leq x\\ p\equiv 1 \pmod c}} \left(\omega_n\left(\frac{p-1}{c}\right)-\log\log n\right)^2=0\left(\frac{\pi(x)}{\varphi(c)}\log\log n\right),$$

where $\varphi(c)$ is the Euler function.

Above two lemmas are easy applications of the Siegel-Walfisz Theorem [3, Satz 8.3].

Proof of the theorem. As in [1] we divide the integers lying between n and $n \exp(h^{-1}(n)\log\log n)$ into two classes. Namely, in the first class we put the integers $b_1, b_2, \cdots, b_y$ having at most $\frac{2}{3}\log\log n$ prime divisors and in the second class the integers $c_1, \cdots, c_z$ having more than $\frac{2}{3}\log\log n$ prime divisors.