174. *Elliptic Modular Surfaces. I*

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Introduction. Let B denote an elliptic surface over a non-singular algebraic curve Δ having a global holomorphic section o. We denote by J and G the functional and homological invariants of B, and by $\mathcal{F}(J, G)$ the family of elliptic surfaces over Δ with the same functional and homological invariants as B. We refer to Kodaira [1] for the general theory of elliptic surfaces. The family $\mathcal{F}(J, G)$, modulo suitable equivalence, is parametrized by the cohomology group $H^1(\Delta, \Omega(B^\#))$ (or by $H^1(\Delta, \Omega(B_0^\#))$), where $B^\#$ (or $B_0^\#$) denotes the group scheme over Δ associated with B (or the connected component of the identity section o in $B^\#$) and where $\Omega(B^\#)$ (or $\Omega(B_0^\#)$) denotes the sheaf of germs of holomorphic sections of $B^\#$ (or $B_0^\#$) over Δ. Moreover the torsion elements in the group $H^1(\Delta, \Omega(B^\#))$ correspond to algebraic surfaces in the family $\mathcal{F}(J, G)$. Now Kodaira raised the question whether or not algebraic surfaces are dense in the family $\mathcal{F}(J, G)$, which has motivated our present work.

In this (and the forthcoming) paper, we consider the special case where B, Δ, J and G are defined in terms of a torsion-free subgroup Γ of finite index of the homogeneous modular group $SL(2, \boldsymbol{Z})$ (see Section 2 for the definition). We write, if necessary, $B_\Gamma, \Delta_\Gamma, \cdots$ for $B, \Delta, \cdots$ to specify the group Γ. The base curve Δ_Γ of B_Γ is the compact Riemann surface associated with the Fuchsian group Γ acting on the upper half plane. The elliptic surface B_Γ will be called the *elliptic modular surface* attached to Γ. The main results can be stated as follows.

Theorem. *Let B denote an elliptic modular surface with the base curve Δ. Then*

(i) *B has only a finite number of global sections over Δ.*

(ii) *The group $H^1(\Delta, \Omega(B_0^\#))$ is isomorphic to a product of a complex torus and a finite group.*

The complex torus in question is an analogue of Shimura's abelian varieties attached to cusp forms of even weights [3]; here it is related to cusp forms of weight 3. We do not know whether or not this complex torus has an structure of an abelian variety. As an immediate consequence of the theorem, we get a partial answer to Kodaira's question: