197. *Necessary and Sufficient Conditions for the Normality of the Product of Two Spaces*

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In this paper we shall present a solution (Theorem) of the problem to find necessary and sufficient conditions for the normality of a product space $X \times Y$.

This fundamental problem has been researched by many mathematicians probably since about the time (1925) when the importance of normal spaces was found by P. Urysohn [1]. Though the problem has been unsettled for a fairly long time, the proof of our Theorem is simple and elementary. Difficulty may have been in the formulation of the theorem, which is natural but apparently pretty different from ones conjectured from known partial solutions.

In this paper a space is, unless otherwise specified, topological.

Let A be a subset of the product space $X \times Y$ of spaces X and Y, then we write for $x \in X$

$$A[x] = \{y \in Y\,;\,(x, y) \in A\}.$$

Definition 1. Let $\mathfrak{F}$ be a family of neighborhoods of $a \in X$, and let $\{A_x\,;\,x \in Z \subset X\}$ any family of subsets A_x of Y, then we write

$$\limsup_{\mathfrak{F}} A_x = \bigcap_{U \in \mathfrak{F}} \overline{\bigcup_{x \in U} A_x},$$

$$c\text{-}\limsup_{\mathfrak{F}} A_x = Y - \limsup_{\mathfrak{F}} (Y - A_x),$$

where the bar means the closure in Y and $x \in U$ does $x \in U \cap Z$.

Hereafter, let us denote by $\mathfrak{N}_a$ for $a \in X$ the neighborhood system of a in X, and we write "$\limsup_a$" instead of "$\limsup_{\mathfrak{N}_a}$". We can easily obtain

Proposition 1. *Let $\mathfrak{M}$ be a neighborhood base of a in X, then*

$$\limsup_a A_x = \limsup_{\mathfrak{M}} A_x,$$

$$c\text{-}\limsup_a A_x = c\text{-}\limsup_{\mathfrak{M}} A_x.$$

Proposition 2. *Let $\{A_x\,;\,x \in Z \subset X\}$ be any family of sets $A_x \subset Y$, and put*

$$(x, A_x) = \{(x, y)\,;\,y \in A_x\},$$

$$A = \bigcup_{x \in Z} (x, A_x),$$

then