185. *On the Univalence of Certain Integral*

By Mamoru NUNOKAWA
Gunma University

(Comm. by Kunihiko KODAIRA, M. J. A., Dec. 12, 1969)

1. Let S be the class of functions $f(z)$ regular, univalent in $|z|<1$ and normalized by $f(0)=0$, $f'(0)=1$. On the other hand, let C, S^* and K be the subclass of S convex, starlike and close-to-convex functions, respectively. In the recent papers, [1], [2], [3, p. 40], [6, 7] and [9], the univalence of the functions

$$g(z)=\int_0^z\left(\frac{f(t)}{t}\right)^\alpha dt \qquad \text{and} \qquad g(z)=\int_0^z(f'(t))^\alpha dt$$

was studied.

2. On the univalence of $g(z)=\int_0^z(f'(t))^\alpha dt$.

Lemma 1. *Let $f(z)$ be regular for $|z|\leqq r$ and $f'(z)\neq 0$ on $|z|=r$. Suppose that on $|z|=r$*

$$\int_0^{2\pi} darg\, df(z)=\int_0^{2\pi}\frac{\partial}{\partial\theta}[\arg zf'(z)]d\theta=\int_0^{2\pi}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)d\theta=2\pi$$

If furthermore

$$\int_{\theta_1}^{\theta_2} darg\, df(z)=\int_{\theta_1}^{\theta_2}\frac{\partial}{\partial\theta}[\arg zf'(z)]d\theta<3\pi \quad for \quad \theta_1<\theta_2$$

or

$$\int_{\theta_1}^{\theta_2} darg\, df(z)=\int_{\theta_1}^{\theta_2}\frac{\partial}{\partial\theta}[\arg zf'(z)]d\theta>-\pi \quad for \quad \theta_1<\theta_2$$

then $f(z)$ is univalent and close-to-convex in $|z|<r$.

We owe this lemma to Umezawa [11] and Reade [8].

Lemma 2. *Let $f(z)=z+\sum_{n=2}^\infty a_nz^n\in K$. Then there is a $\rho<1$ such that for all γ in the interval $\rho<\gamma<1$*

$$\int_{|z|=r} darg\, df(z)=2\pi$$

and

$$3\pi>\int_C darg\, df(z)>-\pi,$$

where C is an arbitrary arc on the boundary $|z|=r$.

We owe this lemma to Umezawa [12, Theorem 1].

Theorem 1. *Let $f(z)=z+\sum_{n=2}^\infty a_nz^n\in K$. Then*

$$g(z)=\int_0^z(f'(t))^\alpha dt$$