184. On the Topological Entropy of a Dynamical System

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§1. **Preliminaries.** Let φ be a homeomorphism from a compact space X onto itself. If α is any open cover of X, we let $N(\alpha)$ be the number of members in a subcover of α of minimal cardinality. As in [1], the limit exists in the following definition:

$$h(\alpha, \varphi)=\lim_{n\to\infty} \frac{1}{n} \log N(\vee_{i=0}^{n-1}\varphi^i\alpha)^{*)}$$

and the topological entropy $h(\varphi)$ of φ is defined as $h(\varphi)=\sup h(\alpha, \varphi)$, where the supremum is taken over all open covers of X.

Let $\{\varphi_t\}$ be a homeomorphic flow on a compact space X. It was conjectured in [1] that

(F) $\qquad h(\varphi_t)=|t|\, h(\varphi_1)$ for all t.**)

In this paper, we will give a proof for (F) under the assumption that X is a compact metric space and that $\{\varphi_t\}$ is a dynamical system: namely,

(C) $\qquad \varphi_t x=\varphi(t, x)$ is continuous in the pair of variables t, x.

For later use, we will mention the well-known properties of the topological entropy [1]:

(1.1) $\qquad$ if $\alpha<\beta$ then $h(\varphi, \alpha)\leqslant h(\varphi, \beta)$

(1.2) $\qquad h(\varphi_k)=|k|\, h(\varphi_1)$ for integer k.

§2. **The theorem.** **Theorem.** If X is a compact mentric space and $\{\varphi_t\}$ is a dynamical system. Then

(F) $\qquad h(\varphi_t)=|t|\, h(\varphi_1)$ for any real t.

Proof. For any pair $\varepsilon_1>\varepsilon_2>0$, let α_i be the set of all open spheres of radius ε_i, $i=1, 2$. Then $\alpha_1<\alpha_2$.

Put $A_t=\{x\,|\,d(\varphi_s x, x)<\varepsilon_1-\varepsilon_2$ for $|s|\leqslant t\}$, where d is the distance of X. Then, by the continuity of $\{\varphi_t\}$, the set A_t is an open set, and $A_t\subset A_{t'}$ if $t>t'$, and moreover, $\bigcup_{t>0} A_t=X$. Thus, by the compactness of X, there exists a positive real number t_0 satisfying $A_{t_0}=X$.

If t, t' are arbitrary pair of positive numbers and T is an arbitrary large positive number, there exist positive integers p, n and m such that $t/p\leqslant t_0$,

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*) As in [1], we write $\alpha\vee\beta=\{U\cap V : U\in\alpha\ V\in\beta\}$ and we write $\alpha>\beta$ to mean that α is a refinement of β.

**) On the measure theoretic entropy (F) is proved in [2]; much simpler proof is given in [3].