183. Elliptic Modular Surfaces. II

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In the first part [6], we have introduced a special class of elliptic surfaces called elliptic modular surfaces. In this part II, we shall indicate the proof of the theorem announced in [6] (Theorems 3.1 and 5.4). A reformulation and a few remarks will be given in Section 6.

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Notation. We use the same notations as in [6]. In particular, Γ always denotes a torsion-free subgroup of finite index of $SL(2, \boldsymbol{Z})$ (except in Remark 6.6).

3. The group of sections. In this section we shall prove

Theorem 3.1. *An elliptic modular surface has only a finite number of sections over the base curve.*

We denote by μ the index of the subgroup $\Gamma\{\pm 1_2\}$ in $SL(2, \boldsymbol{Z})$, and by t_1 (or t_2) the number of cusps of the first (or second) kind; put $t=t_1+t_2$. Then the genus g of the curve $\varDelta=\varDelta_\Gamma$ is given by the formula: $2g-2+t=\mu/6$. The index μ is clearly equal to the order of the meromorphic function J on $\varDelta$, the functional invariant of the elliptic modular surface B_Γ. Hence, from Theorem 12.2 of [1], we can compute the arithmetic and geometric genus of B_Γ.

Lemma 3.2. $p_a=\mu/12+t_2/2-1$,
$p_g=2g-2+t-t_1/2$.

Comparing Lemma 3.2 with Theorem 1.2 and Corollary 1.4, we get

Lemma 3.3. $r=0$ *and* $r'=2p_g$.

Thus the group $H^0(\varDelta, \Omega(B_0^\sharp))$ is of rank 0, i.e., finite. By considering the exact sequence ([1], Section 11)

$$(***) \qquad 0\to\Omega(B_0^\sharp)\to\Omega(B^\sharp)\to Q\to 0,$$

where the quotient Q is a sheaf of finite groups with the support on the finite set $\varDelta-\varDelta'$, we conclude that the group $H^0(\varDelta, \Omega(B^\sharp))$ is also finite, which completes the proof of Theorem 3.1.

Example 3.4. For the elliptic modular surface $B(N)$ for level N ($N\geq 3$) (cf. Example 2.1—where we used the above Lemma 3.3), we can show that the group of sections of $B(N)$ is isomorphic to the finite group $(\boldsymbol{Z}/N\boldsymbol{Z})^2$. Moreover any two distinct sections do not meet each other. When $N=3$, $B(3)$ is a rational surface and the 9 sections are mutually disjoint exceptional curves of the first kind (cf. [6a]. p. 464).