14. Some Cross Norms which are not Uniformly Cross

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It is known that all C^*-norms in the algebraic tensor product of two C^*-algebras are cross. We shall show that no C^*-norms are uniformly cross in R. Schatten's sense [4] in the algebraic tensor product of two non-abelian C^*-algebras if one of them has an anti-*-automorphism of period two. Also, some examples will show that actually there are *not uniformly cross* C^*-norms. This fact may be felt strange at a glance and will be worth researching.

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1. Preliminaries. Let E and F be Banach spaces, $E\odot F$ the algebraic tensor product of E and F, $\|\ \ \|_\beta$ a norm in $E\odot F$ and $E\hat{\otimes}_\beta F$ the tensor product of E and F with respect to $\|\ \ \|_\beta$, that is, the completion of $E\odot F$ with respect to $\|\ \ \|_\beta$.

If $\|\ \ \|_\beta$ satisfies the relation

$$\|u\otimes v\|_\beta=\|u\|\,\|v\| \text{ for each } u\in E \text{ and } v\in F,$$

then it is said to be cross; also, if $\|\ \ \|_\beta$ is cross and if for each pair of bounded linear operators ρ on E and σ on F, the relation

$$\|\Sigma_i\rho(u_i)\otimes\sigma(v_i)\|_\beta\leqq\|\rho\|\,\|\sigma\|\,\|\Sigma_iu_i\otimes v_i\|_\beta \quad \text{for each } \Sigma_iu_i\otimes v_i\in E\odot F$$

is satisfied, in other words, the operator norm of the linear operator

$$(\rho\otimes\sigma)(\Sigma_iu_i\otimes v_i)=\Sigma_i\rho(u_i)\otimes\sigma(v_i)$$

on $E\odot F$ is finite and not greater than $\|\rho\|\,\|\sigma\|$, then $\|\ \ \|_\beta$ is said to be uniformly cross (see [4], V and VI in pp. 28–29).

Let A and B be C^*-algebras. A norm $\|\ \ \|_\beta$ in the algebraic tensor product $A\odot B$ of A and B is called a C^*-norm if $\|t^*t\|_\beta=\|t\|_\beta^2$ for all $t\in A\odot B$. It is obvious that if $\|\ \ \|_\beta$ is a C^*-norm then $A\hat{\otimes}_\beta B$ becomes a C^*-algebra in the usual way.

The most natural C^*-norm in $A\odot B$ is the α-norm $\|\ \ \|_\alpha$ defined by

$$\|\Sigma_ia_i\otimes b_i\|_\alpha=\|\Sigma_i\pi_1(a_i)\otimes\pi_2(b_i)\| \qquad \text{for } \Sigma_ia_i\otimes b_i\in A\odot B,$$

using arbitrarily chosen faithful *-representations π_1 of A and π_2 of B, where the right side means the operator norm of the operator $\Sigma_i\pi_1(a_i)\otimes\pi_2(b_i)$ on the tensor product $H_1\otimes H_2$ of the representation Hilbert spaces H_1 of π_1 and H_2 of π_2 (see [6], [7]). Another C^*-norm in $A\odot B$ is referred to [1] and [3].

The reason why a C^*-norm $\|\ \ \|_\beta$ is cross lies in the facts that the α-norm is cross, that $\|t\|_\alpha\leqq\|t\|_\beta$ (Theorem 2 in [5]) and that $\|x\otimes y\|_\beta$