7. *Sufficient Conditions for the Normality of the Product of Two Spaces*

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In this note, an application of [2] and [3] is shown which deduces known best results giving sufficient conditions stated in the title. An example is presented which suggests the reason why we have generally had only few necessary conditions of each of X and Y separately for the normality of $X\times Y$.

Spaces in this note are normal (and so Hausdorff). We use notations and terminologies in [2] and [3].

Theorem 1. *Suppose that X is normal and $\mathfrak{m}$-paracompact, Y is normal and has an open base for closed sets of power $\leqq \mathfrak{m}$, and Y is upper compact at X, then $X\times Y$ is normal.*

Proof. Let $\{A_x\subset Y; x\in X\}$ and $\{B_x\subset Y; x\in X\}$ be arbitrary families with

$$(1)\qquad \limsup_a A_x \cap \limsup_a B_x=\emptyset$$

for any $a\in X$. Let $\{E_\lambda; \lambda\in\Lambda\}$ be an open base for closed sets in Y, including the empty set and Y, with $\|\Lambda\|\leqq\mathfrak{m}$. Put $A=\bigcup_{x\in X}(x, A_x)$ and $B=\bigcup_{x\in X}(x, B_x)$, then

$$O_\lambda=\{x; \bar{A}[x]\subset E_\lambda\}\cap\{x; \bar{B}[x]\subset \mathcal{C}\overline{E_\lambda}\}$$

is open by Proposition 3 in [3]. Since, for any $x\in X$, $\bar{A}[x]$ and $\bar{B}[x]$ are disjoint closed sets of the normal space Y, there is an E_λ such that $\bar{A}[x]\subset E_\lambda$ and $\overline{E_\lambda}\subset\mathcal{C}\bar{B}[x]$, so $\{O_\lambda; \lambda\in\Lambda\}$ is an open cover of X with power $\leqq\mathfrak{m}$, and there is a locally finite open refinement $\{Q_\lambda; \lambda\in\Lambda\}$ with $\overline{Q_\lambda}\subset O_\lambda$ for every $\lambda\in\Lambda$. Let us put

$$G_x=\bigcup_{\overline{Q_\lambda}\ni x} E_\lambda.$$

Take a point a, then there are Q_{λ_0} including a and $U_0\in\mathfrak{N}_a$ such that U_0 meets only a finite number of $\overline{Q_\lambda}$ and $U_0\subset Q_{\lambda_0}$. $\{G_x; x\in U_0\}$ consists of finitely many different open sets, and if $x\in U_0$, then $x\in Q_{\lambda_0}$ and $G_x\supset E_{\lambda_0}\supset\bar{A}[a]\supset A_a$, so we have

$$c\text{-}\limsup_a G_x\supset(\bigcap_{x\in U_0} G_x)^0=\bigcap_{x\in U_0} G_x\supset A_a.$$

While, there is $U_1\in\mathfrak{N}_a$ such that $U_1\subset U_0$ and $U_1\cap\overline{Q_\lambda}\neq\emptyset$ implies $\overline{Q_\lambda}\ni a$, $O_\lambda\ni a$ and $B_a\cap\overline{E_\lambda}=\emptyset$. Let only $\overline{Q_{\lambda_1}}, \cdots, \overline{Q_{\lambda_n}}$ meet U_1, then $x\in U_1$ follows