23. Every C-Symmetric Banach $*$-Algebra is Symmetric[1)]

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Since Gelfand and Naimark [1] had conjectured the symmetry of B^*-algebras, I. Kaplansky raised the more general question: is a C-symmetric Banach $*$-algebra symmetric?, and it remained open during the past twenty years. One knew that an intrinsic key to this problem is to prove that the sum of positive elements is also positive (for B^*-algebras, see [2], [5]). Recently we have proved in [9] that in a Banach $*$-algebra with the norm condition $\alpha\|x^*\|\|x\|\leq\|x^*x\|$ $(\alpha>0)$, the positive elements form a positive cone, and then it turned out that the method employed there can be applied for C-symmetric Banach $*$-algebras by a slight modification. Meantime, we have heard that S. Shirali and J. Ford [8] have solved Kaplansky's problem in the affirmative, that is, the following result has been established.

Theorem. *A C-symmetric Banach $*$-algebra is necessarily symmetric.*

In this paper we will supply a technically simple and possibly quick proof of the theorem by an adequate improvement of our previous work [9].

1. Let us recall that a Banach $*$-algebra A is symmetric if x^*x is quasi-regular for all x in A; it is C-symmetric if every closed commutative $*$-subalgebra is symmetric. In case A has a unit e, the symmetry means that $e+x^*x$ is invertible for every x in A and therefore the C-symmetry means that $e+x^*x$ is invertible for every normal element x (i.e., $x^*x=xx^*$) in A. Throughout this paper we shall mainly concern a (complex) Banach $*$-algebra A with unit e. We denote by $\sigma(x)$ the spectrum of an element x in A; a self-adjoint element h in A is said to be positive (strictly positive) if $\sigma(x)\subset[0,\infty)$ $((0,\infty))$, respectively, and then for self-adjoint elements h, k in A, we understand the symbol $h\leq k$ (or $h<k$) as usual. For a normal element x in A, $A(x)$ always means a maximal commutative $*$-subalgebra of A containing x, and we should recall that $A(x)$ is automatically closed. In the proof of the theorem, the fact that a strictly positive element in A has a (strictly) positive square root will play a relevant role (cf. [3], [10]).

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