47. *On Homogeneous Complex Manifolds with Negative Definite Canonical Hermitian Form*

By Hirohiko SHIMA
Osaka University

(Comm. by Kenjiro SHODA, M. J. A., March 12, 1970)

Throughout this note, G denotes a connected Lie group and K is a closed subgroup of G. We assume that G acts effectively on the homogeneous space G/K. Suppose that G/K carries a G-invariant complex structure I and a G-invariant volume element v. Then we may define canonical hermitian form associated to I and v [2].

Theorem. *Let G/K be a homogeneous complex manifold with a G-invariant volume element. If the canonical hermitian form h of G/K is negative definite, then G is a semisimple Lie group.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of all left invariant vector fields on G and $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to K. We denote by I the G-invariant complex structure tensor on G/K. Let π_e be the differential of the canonical projection π from G onto G/K at the identity e and let $I_{e'}$ (resp. X_e) be the value of I (resp. $X \in \mathfrak{g}$) at $\pi(e)=e'$ (resp. e). Koszul [2] proved that there exists a linear endomorphism J of $\mathfrak{g}$ such that for $X, Y \in \mathfrak{g}$ and $W \in \mathfrak{k}$

$$\pi_e(JX)_e = I_{e'}(\pi_e X_e) \tag{1}$$

$$J\mathfrak{k} \subset \mathfrak{k} \tag{2}$$

$$J^2X \equiv -X \bmod \mathfrak{k} \tag{3}$$

$$[JX, W] \equiv J[X, W] \bmod \mathfrak{k} \tag{4}$$

$$[JX, JY] \equiv J[JX, Y] + J[X, JY] + [X, Y] \bmod \mathfrak{k} \tag{5}$$

Moreover, the canonical hermitian form h of G/K associated to the G-invariant volume element is expressed as follows. Putting

$$\eta = \pi^* h,$$

$$\eta(X, Y) = \frac{1}{2}\psi([JX, Y]) \tag{6}$$

for $X, Y \in \mathfrak{g}$, where $\psi(X) =$ trace of $(ad(JX) - Jad(X))$ on $\mathfrak{g}/\mathfrak{k}$ for $X \in \mathfrak{g}$. As h is assumed to be negative definite, $\eta(X, X) \leqq 0$ for any $X \in \mathfrak{g}$, and $\eta(X, X) = 0$ if and only if $X \in \mathfrak{k}$. Therefore, putting $\omega = -\psi$, $(\mathfrak{g}, \mathfrak{k}, J, \omega)$ is a j-algebra in the sense of E. B. Vinberg, S. G. Gindikin and I. I. Pjateckii-Šapiro [4].

Now suppose that $\mathfrak{g}$ is not a semisimple Lie algebra. Then there is a non-zero commutative ideal $\mathfrak{r}$ of $\mathfrak{g}$. Consider the J-invariant subalgebra