82. *Notes on Modules. III*

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(Comm. by Kinjirô KUNUGI, M. J. A., April 13, 1970)

In this paper we discuss the Kertész' radical for modules, and among other we show that this radical fails to be a ring radical in the sense of Amitsur and Kurosh. We refer yet concerning this topic to our earlier papers [6], [7].

Following Kertész [3], for an arbitrary ring A and for any right A-module M, we consider the set

(1) $$K(M)=\{X; X\in M, \quad XA\subseteq\Phi(M)\}$$

where $\Phi(M)$ denotes the Frattini A-submodule of M. (That is, $\Phi(M)$ is the intersection of all maximal submodules of M, and $\Phi(M)=M$ for modules M having no maximal A-submodules.) Obviously, $K(M)$ is an A-submodule of M. Calling an A-submodule N of M homoperfect, if

(2) $$MA+N=M$$

holds, then (1) implies by Kertész [3], that $K(M)$ coincides with the intersection of all homoperfect maximal A-submodules of M

Example. For a prime number p let A be the ring generated by the 3×3 matrices over the field of p elements:

(3) $$x=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix},\qquad y=\begin{bmatrix}1&0&0\\0&1&0\\0&1&0\end{bmatrix}$$

Then A is a noncommutative ring with p^2 elements and with the multiplication:

(4)

	x	y
x	0	x
y	0	y

By a routine calculation it can be verified that the principal right ideal $(y)_r$ of A is a homoperfect maximal right ideal, but $(y)_r$ is neither modular, nor quasimodular in A.

Furthermore, for the Kertész radical $K_r(A)$ of the A-right module A, one has by

(5) $$(x)_r\cap(y)_r=0$$

obviously $K_r(A)=0$, being also $(x)_r$ homoperfect and maximal in A. The Jacobson radical $F(A)$ of A now coincides with $(x)_l=K_l(A)$, denoting $K_l(A)$ the left-right dual of $K_r(A)$

Therefore, this ring A has the property, that

(6) $$0=K_r(A)\neq K_l(A)=F(A)$$