81. *Notes on Modules. II*

By Ferenc A. SZÁSZ

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Generalizing a well known important result (cf. Jacobson [1], Chapter IV, p. 93) for vector spaces, in our paper all twosided ideals of the total endomorphism ring $E(M)$ of a homogeneous completely reducible module M over an arbitrary ring A are determined. Our result is an English version of the earlier paper of the author [2].

Theorem. *Let $E(M)$ be the total endomorphism ring of a homogeneous completely reducible right A-module M over an arbitrary ring A. Then for every nonzero twosided ideal $\mathcal{J}$ of $E(M)$ there exists an infinite cardinality $\mathfrak{M}$ such that $\mathcal{J}$ coincides with the set of all endomorphisms γ of M with* rang $\gamma M < \mathfrak{M}$

Proof. We assume that rang $M \geqq \aleph_0$ over A, being $E(M)$ a simple total matrix ring over a division ring for the particular case

$$\text{rang } M < \aleph_0.$$

1. Firstly we assert that if $\mathcal{J}$ is a twosided ideal of $E(M)$ with $\gamma_2 \in \mathcal{J}$ and

$$(1) \qquad \text{rang } \gamma_1 M \leqq \text{rang } \gamma_2 M$$

for an arbitrary $\gamma_1 \in E(M)$, then $\gamma_1 \in \mathcal{J}$

Namely, for $i=1$ and $i=2$ let N_i be the kernel of the endomorphism γ_i in M. Then there exists a completely reducible submodule K_i of M with $M = N_i \oplus K_i$. Then (1) implies

$$(2) \qquad \text{rang } K_1 \leqq \text{rang } K_2$$

If $K_i = \sum_{(i)} \oplus \{k_{\alpha_j}\}$, then by (2) and by the fact that M is homogeneous, there exists an endomorphism $\delta_1 \in E(M)$ such that holds

$$(3) \qquad \delta_1 \underset{(1)}{k_{\alpha_1}} = \underset{(2)}{k_{\alpha'_1}} \quad \text{and} \quad \delta_1 N_1 = 0$$

Here α'_1 denotes an uniquely determined index α_2 from Γ_2, and for $\alpha_1 \neq \beta_1$ one has obviously $\alpha'_1 \neq \beta'_1$ $(\alpha_1, \beta_1 \in \Gamma_1;\ \alpha_2, \beta_2 \in \Gamma_2$, being Γ_2 the set of indices of fixed basis elements of $K_i)$. Consequently, the restriction of δ_1 on $\delta_1 K_1$ has an inverse element δ_1^{-1}.

From an assumed linear connection

$$(4) \qquad \sum_{j=1}^{n} \gamma_2 \delta_1 \underset{(1)}{k_{\alpha_j}} a_j = 0 \quad (a_j \in A)$$

follows $\gamma_2 k^* = 0$ for the element

$$k^* = \sum_{j=1}^{n} \delta_1 \underset{(1)}{k_{\alpha_j}} a_j \in K_2$$

Therefore $k^* \in N_2 \cap K_2$, and $k^* = 0$. There exists an inverse element