77. *On Nest Algebras of Operators*

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1. In this paper we study certain algebras of operators termed 'nest algebras', which were introduced by J. R. Ringrose [3]. Our main results (Theorems 4 and 5) are concerned with characterizations of such algebras, and consequently it is proved that each weakly closed maximal triangular operator algebra is hyperreducible.

Throughout this paper the terms *Hilbert space*, *subspace*, *operator*, *projection* are used to mean *complex* Hilbert space, *closed linear* subspace, *bounded linear* operator, *orthogonal* projection, respectively. Given a subspace $\mathfrak{M}$ of a Hilbert space $\mathfrak{H}$, we shall write $P_{\mathfrak{M}}$ for the projection from $\mathfrak{H}$ onto $\mathfrak{M}$, and $\mathfrak{H}\ominus\mathfrak{M}$ for the orthogonal complement of $\mathfrak{M}$ in $\mathfrak{H}$. If $\{\mathfrak{M}_\alpha\}$ is a collection of subspaces of $\mathfrak{H}$, then the smallest subspace which contains each $\mathfrak{M}_\alpha$ will be denoted by $\vee\mathfrak{M}_\alpha$, and the largest subspace contained in each $\mathfrak{M}_\alpha$ will be denoted by $\wedge\mathfrak{M}_\alpha(=\cap\mathfrak{M}_\alpha)$. Set inclusion in the wide sense will be denoted by the symbol '$\subseteq$', and we reserve '$\subset$' for proper inclusion.

The class of all operators from a Hilbert space $\mathfrak{H}$ into itself will be denoted by $\mathcal{L}(\mathfrak{H})$. By an *algebra of operators* on $\mathfrak{H}$ we shall mean a subset $\mathcal{A}$ of $\mathcal{L}(\mathfrak{H})$ such that, if λ is a complex number and $A, B \in \mathcal{A}$, then $A+B, AB, \lambda A \in \mathcal{A}$. A self-adjoint algebra of operators will be termed a $*$-algebra.

2. Following J. R. Ringrose, a family $\mathcal{N}$ of subspaces of a Hilbert space $\mathfrak{H}$ will be called a *nest* if it is totally ordered by the inclusion relation $\subseteq$; $\mathcal{N}$ will be called a *complete* nest if, further,

(i) $(0), \mathfrak{H} \in \mathcal{N}$;

(ii) given any subnest $\mathcal{N}_0$ of $\mathcal{N}$, the subspaces $\bigwedge_{\mathfrak{M}\in\mathcal{N}_0} \mathfrak{M}$, $\bigvee_{\mathfrak{M}\in\mathcal{N}_0} \mathfrak{M}$ are both members of $\mathcal{N}$.

Given a complete nest $\mathcal{N}$ and a non-zero subspace $\mathfrak{M}$ in $\mathcal{N}$, we define

$$\mathfrak{M}_- = \vee\{\mathfrak{N} \mid \mathfrak{N} \in \mathcal{N}, \mathfrak{N}\subset\mathfrak{M}\}.$$

Clearly $\mathfrak{M}_- \in \mathcal{N}$.

If $\mathcal{N}$ is a complete nest of subspaces of a Hilbert space $\mathfrak{H}$, then the *nest algebra* $\mathcal{A}_{\mathcal{N}}$ associated with $\mathcal{N}$ is defined to be the class of all operators on $\mathfrak{H}$ which leave invariant each subspace in $\mathcal{N}$. Clearly $\mathcal{A}_{\mathcal{N}}$ is a weakly closed subalgebra of $\mathcal{L}(\mathfrak{H})$.

The following lemma, included here for the sake of completeness,