101. A Remark on the S-Equation for Branching Processes

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Let $\{x_t, \mathcal{B}_t\}$ be a right-continuous strong Markov process on a metric space D. Assume $\{\phi_t\}$ is a finite increasing additive functional of $\{x_t\}$, and let $\{\pi_n(a, E)\}$ be a series of substochastic kernels for $a \in D$, $E \subseteq D^n$ satisfying $\sum \pi_n(a, D^n)=1$.[1] Consider the branching process $\{z_t\}$ in D (actually in $X=\bigcup_0^\infty D^n$) determined by $\{x_t\}$, the branching rate $d\phi_t$ $\left(\text{thus if } \phi_t=\int_0^t V(x_s)d_s\text{, an individual particle branches with probability } V(x_t)dt \text{ in time } dt\right)$ and position distributions $\pi_n(x_t, E)$ of the offspring of a particle which does branch. (See [1]-[5]; we use the notation of [4].) The transition function $\bar{P}(t, x, E)$ of $\{z_t\}$ in X can be determined from the transition function $P(t, a, A)$ of $\{x_t\}$ by either a linear equation in X or a non-linear equation in D. The linear equation is the so-called "M-equation".

$$(1)\qquad \bar{T}_t h(x)=E_x(h(z_t)\chi_{[\beta>t]}) + E_x\left(\chi_{[\beta\le t]}\int_X \bar{T}_{t-\beta}h(y)\mu(w, dy)\right)$$

for bounded Borel functions $h(x)$ on X, where $\bar{T}_t h(x)=\int h(y)\bar{P}(t, x, dy)$ and β is the first branching time ($P_a(\beta>t/\mathcal{B}_\infty)=\exp(-\phi_t)$ in D). Alternately, for $a \in D$, we have the "S-equation" ([5])

$$(2)\qquad \bar{T}_t\hat{f}(a)=E_a(f(x_t)\chi_{[\beta>t]}) + E_a\left(\chi_{[\beta\le t]}\sum_0^\infty \int_{D^r}\prod_1^r \bar{T}_{t-\beta}\hat{f}(b_i)\pi_r(x_\beta, db)\right)$$

where $f(a)$ is a Borel function on D, $|f(a)|\le 1$, and $\hat{f}(x)=\prod_1^r f(a_i)$ for $x=(a_1, a_2, \cdots, a_r)$, $f(\partial)=1$. In particular, if new particles are always born at the same location where their parent branches, the non-linearity in (2) is of power series type. As is proven in [2, III], the semi-group $\{\bar{T}_t\}$ can be obtained from either equation.[2] I.e., if $h(x)\ge 0$ (or $f(a)\ge 0$), then $\bar{T}_t h(x)$ (or $\bar{T}_t\hat{f}(a)$) is the minimal non-

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1) Here D^n is the usual n-fold Cartesian product of D with itself, and $D^0=\{\partial\}$, $\partial \notin D$, where $\pi_0(a, \{\partial\})=\pi_0(a)$ refers to x_t dying childless.

2) More exactly, in the case of (2), only those properties of the model which are permutation invariant; see the remark.