97. *Note on the Lexicographic Product of Ordered Semigroups*

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A semigroup S with a simple order $\leqq$ is called a *left* [*right*] *ordered semigroup* if it satisfies the condition that

for every $x, y, z \in S$, $x \leqq y$ *implies* $zx \leqq zy$ [$xz \leqq yz$].

S is called an ordered semigroup if it is a left and right ordered semigroup. Let $\{S_\alpha; \alpha \in A\}$ be a collection of semigroups, each of which has a simple order and let the index set A be a well-ordered set. The direct product semigroup $\prod_{\alpha \in A} S_\alpha$ is called the *lexicographic product* of $\{S_\alpha; \alpha \in A\}$ if the simple order $\leqq$ in $\prod_{\alpha \in A} S_\alpha$ is defined by

$(a_1, \cdots a_\alpha, \cdots) < (b_1, \cdots, b_\alpha, \cdots)$ *if and only if there exists an element* $\alpha \in A$ *such that, for every* $\gamma \in A$ *with* $\gamma < \alpha$, $a_\gamma = b_\gamma$ *and moreover that* $a_\alpha < b_\alpha$.

The purpose of this note is to give a condition in order that the lexicographic product of a well-ordered collection of ordered semigroups is an ordered semigroup.

A semigroup S is called *left* [*right*] *condensed* if, for every $s \in S$, sS [Ss] is a one-element set.

Lemma 1. *Let S be a left condensed semigroup. Then there exist a partition of S into $\{T_\lambda; \lambda \in \Lambda\}$ and, for each $\lambda \in \Lambda$, an element $z_\lambda \in T_\lambda$ such that z_λ is a left zero of the semigroup S and that, for every $x_\lambda \in T_\lambda$, $x_\lambda S = z_\lambda$.*

Proof. Let S be a left condensed semigroup. For $a, b \in S$, we define $a \sim b$ if and only if $aS = bS$. Then the relation $\sim$ is an equivalence relation. Hence the set of equivalence classes $\{T_\lambda; \lambda \in \Lambda\}$ forms a partition of S. By definition, for each T_λ, there corresponds an element $z_\lambda \in S$ such that $x_\lambda S = z_\lambda$ for every $x_\lambda \in T_\lambda$. Hence

$$z_\lambda S = x_\lambda S^2 = z_\lambda$$

and so z_λ is a left zero of S and moreover $z_\lambda \in T_\lambda$.

Lemma 2. *A semigroup S is left condensed and left cancellative if and only if S consists of one element.*

Proof. Let S be a left condensed and left cancellative semigroup and let $x, y \in S$. Since S is left condensed, we have $x^2 = xy$ and then, since S is left cancellative, $x = y$. Hence S consists of one element. The converse part is trivial.

Lemma 3. *A semigroup S is left condensed and right cancellative*