92. *On Cubic Galois Extensions of* $Q(\sqrt{-3})$

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Let k be the field $\mathbf{Q}(\sqrt{-3})$ and let K be the field $k(\sqrt[3]{A})$ for some element A of k. In this paper, we shall determine in Theorem 1 a basis of integers of K and determine in Theorem 2 the genus field of K with respect to k and determine in Theorem 3 whether the class number of K is a multiple of 3 or not

1. A basis of integers.

Let O_k be the ring of integers of $k=\mathbf{Q}(\sqrt{-3})$. Any cubic galois extension K over k can be written as $k(\sqrt[3]{A})$, where $A\in O_k$, $A\neq 1$, is without cubic factors and, without loss of generality, we may assume that $A=fg^2$, f and g being integers of k having no square factors and $f\not\equiv -1$, $g\not\equiv -1 \pmod{\sqrt{-3}}$. Put $A^*=f^2g$, $\theta=\sqrt[3]{A}$, $\theta^*=\theta^2/g=\sqrt[3]{A^*}$ and $O_K=$the ring of integers of K. By the relation $\theta^2=g\theta^*$, every element of K can be expressed in the form $\alpha+\beta\theta+\gamma\theta^*$, $(\alpha,\beta,\gamma\in k)$. Let $\omega=\alpha+\beta\theta+\gamma\theta^*$ be an element of O_K and ω',ω'' be its conjugates over k. It can be easily veryfied that:

(1) $\omega+\omega'+\omega''=3\alpha$,

(2) $\omega\omega'+\omega'\omega''+\omega''\omega=3\alpha^2-3\beta\gamma fg$,

(3) $\omega\omega'\omega''=\alpha^3+\beta^3A+\gamma^3A^*-3\alpha\beta\gamma fg$.

As ω is an integer, 3α and

$$(3\beta)^3A\cdot(3\gamma)^3A^*=(9\beta\gamma fg)^3,$$

$$(3\beta)^3A+(3\gamma)^3A^*=27(\alpha^3+\beta^3A+\gamma^3A^*-3\alpha\beta\gamma fg)-(3\alpha)^3+3\cdot 3\alpha\cdot 9\beta\gamma fg$$

are integers of k. Since A and A^* contain no cubic factors, 3β and 3γ are integers of k. Put $3\alpha=a$, $3\beta=b$ and $3\gamma=c$. Then $\omega=(a+b\theta+c\theta^*)/3$, $(a,b,c\in O_k)$. From (2) and (3), these coefficients must satisfy the congruences:

(4) $a^2-bcfg\equiv 0 \pmod 3$,

(5) $a^3+b^3A+c^3A^*-3abcfg\equiv 0 \pmod{27}$.

We shall next determine a basis of O_K as O_k-module. When $\omega_1=1$, $\omega_2=(a_2+b_2\theta)/3$ and $\omega_3=(a_3+b_3\theta+c_3\theta^*)/3$ are elements of O_K such that:

$$\min\{|b|\,;\,O_K\ni(a+b\theta)/3,\ O_k\ni a,b,\ b\neq 0\}=|b_2|,$$

$$\min\{|c|\,;\,O_K\ni(a+b\theta+c\theta^*)/3,\ O_k\ni a,b,c,\ c\neq 0\}=|c_3|,$$

then $\omega_1,\omega_2,\omega_3$ is a basis of O_K as O_k-module, since O_k is Euclidean.

$(a+b\theta)/3$ is an element of O_K if and only if