194. Dimension of Dispersed Spaces

By Keiô NAGAMI

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Telgársky [5] showed that if X is a paracompact dispersed space, then $\operatorname{ind} X = \dim X = \operatorname{Ind} X = 0$. In this paper we consider the equalities between dimension functions defined on hereditarily paracompact spaces which are dispersed by some classes of spaces. All spaces in this paper are Hausdorff.

Let P be a property such that if a space X has P, then each closed subspace of X has P too. P need not be a topological one. Let $\mathcal{C}$ be the class of all spaces with P. A space X is said to be dispersed by $\mathcal{C}$, to be $\mathcal{C}$-dispersed or to be P-dispersed, if each non-empty closed set of X contains a point x one of whose relative neighborhoods is an element of $\mathcal{C}$. Let Y be a subset of X and Y' the set of all points y in Y one of whose relative neighborhoods is an element of $\mathcal{C}$. Set $Y^{(0)} = Y$, $Y^{(1)} = Y - Y'$ and $Y^{(\alpha)} = \cap \{(Y^{(\beta)})^{(1)} : \beta < \alpha\}$ for an ordinal $\alpha > 0$. Each $X^{(\alpha)}$ is closed. X is $\mathcal{C}$-dispersed if and only if $X^{(\gamma)} = \emptyset$ for some ordinal γ. If X is $\mathcal{C}$-dispersed, then an ordinal-valued function d on X is defined: $d(x) = \alpha$ if and only if $x \in X^{(\alpha)} - X^{(\alpha+1)}$. Let $d(X)$ denote the minimal ordinal α such that $X^{(\alpha)} = \emptyset$.

Theorem 1. *Let X be a hereditarily paracompact space. Then the following are true.*

i) *If X is metric-dispersed, then* $\dim X = \operatorname{Ind} X$.

ii) *If X is separable-metric-dispersed, then* $\operatorname{ind} X = \dim X = \operatorname{Ind} X$.

Proof (by transfinite induction on $d(X)$). Consider the case i). Put the induction assumption that the assertion is true for each hereditarily paracompact space Y with $d(Y) < d(X)$. When $d(X) = 1$, X is locally metric. Hence the whole X is metric by its paracompactness and the equality $\dim X = \operatorname{Ind} X$ is assured by well known Katětov-Morita's theorem. When $d(X) = \alpha + 1$ and $\alpha > 0$, then $(X - X^{(\alpha)})^{(\alpha)} = \emptyset$. Thus $d(X - X^{(\alpha)}) \leq \alpha$ and $\dim (X - X^{(\alpha)}) = \operatorname{Ind} (X - X^{(\alpha)})$ by the induction assumption. Since $\dim X = \max \{\dim X^{(\alpha)}, \dim (X - X^{(\alpha)})\}$ (cf. e.g. Nagami [3, Theorem 9-11]) and $\operatorname{Ind} X = \max \{\operatorname{Ind} X^{(\alpha)}, \operatorname{Ind} (X - X^{(\alpha)})\}$ (cf. Dowker [1, Theorem 3]), we have $\dim X = \operatorname{Ind} X$. When $d(X)$ is the limit ordinal, for each point x of X, $d(x) + 1 < d(X)$. Set $V(x) = X - X^{(d(x)+1)}$. Then $V(x)$ is an open neighborhood of x with $V(x)^{d(x)} = \emptyset$. Hence $\dim V(x) = \operatorname{Ind} V(x)$ by the induction assumption. Since $\dim X = \sup \{\dim V(x) : x \in X\}$ (cf. e.g. Dowker [2, Theorem 3.3]) and $\operatorname{Ind} X$