11. On the Sum $\sum_{n\leq x}\frac{\tilde{n}}{n^2}$

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The purpose of this note is to present an asymptotic formula for the sum described in the title, where we denote by $\tilde{n}$, for every positive integer n, the maximal square-free divisor of n. We shall prove that

$$(1)\qquad \sum_{n\leq x}\frac{\tilde{n}}{n^2}=A\log x+B+O\left(\frac{\log 2x}{\sqrt{x}}\right)\qquad (x>1),$$

where A and B are constants given by

$$A=\prod_p\left(1-\frac{1}{p(p+1)}\right)$$

(the product being taken over all primes p) and

$$B=\frac{\pi^2}{6}c_3-\frac{6}{\pi^2}A\sum_{m=1}^{\infty}\frac{\log m}{m^2}$$

with the constant c_3 determined by (7), (5) and (3).

We note that the asymptotic formula (1) will immediately give a solution to a problem posed by D. Suryanarayana.*)

1. In this paragraph, t denotes an arbitrary real number >1 and k a fixed square-free integer >0. As usual, we denote by $\varphi(k)$ the Euler totient function of k, by $\sigma(k)$ the sum of all positive divisors of k, and by $v(k)$ the number of distinct prime divisors of k. Also, O-constants are all absolute.

It is well known that

$$\sum_{m\leq t}\frac{1}{m}=\log t+C+O\left(\frac{1}{t}\right),$$

where C is the Euler constant. Using this and the well-known property of the Möbius function $\mu(n)$, namely, $\sum_{d|n}\mu(d)=1$ for $n=1$ and $=0$ for $n>1$, we find easily

$$(2)\qquad \sum_{\substack{m\leq t\\(m,k)=1}}\frac{1}{m}=\frac{\varphi(k)}{k}\log t+c_1(k)+O\left(\frac{2^{v(k)}}{t}\right)$$

with

$$(3)\qquad c_1(k)=C\frac{\varphi(k)}{k}-\sum_{d|k}\frac{\mu(d)\log d}{d}=O(\text{loglog }3k).$$

Next, it will follow at once from (2) that

*) Cf. Bull. Amer. Math. Soc., **76**, 976–977 (1970): Problem 17 (2).