8. A Note for Knots and Flows on 3-manifolds

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H. Seifert shows in [1] (Satz 11) that for any torus knot k in the 3-sphere S^3 there is a flow on S^3 with k as an orbit, and conversely, that if a homotopy 3-sphere Σ^3 admits a flow on it so that all orbits are closed then $\Sigma^3=S^3$ and each orbit is a torus knot.

Here, we consider the following question: For any knot k in S^3 does there exist a non-singular flow on S^3 having k as an orbit, allowing for the flow having non-closed orbits? In this paper, we give an affirmative answer to this question.

Manifolds and maps, etc in this paper are assumed to be smooth (C^∞-) ones. A *flow* on a manifold M is a 1-parameter group of transformations $\phi: R\times M\to M$ (R, the real numbers). $x\in M$ is said to be a *singular point* if $\phi(t,x)=x$ for all $t\in R$. ϕ is said to be *non-singular* if there is no singular point. An *orbit* of ϕ passing x is a subset $\{\phi(t,x)\mid t\in R\}$. If there is $t\neq 0$ such that $\phi(t,x)=x$, the orbit is said to be *closed*.

Let f be a map of S^1 into a space M and $p: R\to S^1$ be the usual universal covering defined by $t\mapsto e^{2\pi ti}$, then we shall denote $f\circ p=\bar{f}$.

Theorem. *Let M be an orientable closed 3-manifold and $f: S^1\to M$ be an embedding. Then, there exist a flow $\phi: R\times M\to M$ and $x\in M$ such that $\phi(t,x)=\bar{f}(t)$ for all $t\in R$.*

Proof. Denote the tangent bundle of M by $T(M)$. Since, by [2] (Satz 21), M is parallelizable, we may assume $T(M)=M\times R^3$. Consider the $(R^3-\{0\})$-bundle $T(M)$, $\xi: M\times(R^3-\{0\})\to M$ over M associated to tangent bundle. We define a map $g: f(S^1)\to T(M)$ as follows: for $x\in f(S^1)$, $g(x)=d\bar{f}/dt(t)$ where t is any number such that $\bar{f}(t)=x$. g is well-defined. Since f is an embedding, g is a cross-section of ξ over $f(S^1)$. We will extend g to a cross-section of ξ over M.

We may take a tubular neighborhood U of $f(S^1)$ coordinated as follows;

$$U=\{(x,r,\theta)\mid x\in f(S^1),\quad 0\leqq r\leqq 1,\quad 0\leqq\theta<2\pi\}$$

with

$$(x,0,\theta)=(x,0,0)\quad\text{for all } x \text{ and } \theta.$$

Since $\pi_1(R^3-\{0\})\cong\pi_1(S^2)=0$, we have a homotopy F of $q\circ g$ as follows, where q is the projection into the second factor $M\times(R^3-\{0\})\to R^3-\{0\}$:

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