42. On Strongly Regular Ring

By Mitsuyoshi SOGAWA
Kobe University, Kobe

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In this Note we shall say that the ring A is *regular* if, for every element a of A, there exists an element x in A such that $a=axa$, and A is *strongly regular* if, for every element a of A, there exists an element x in A such that $a=a^2x$, the following theorem is an answer of S. Lajos conjecture.

Theorem. *An associative ring A is strongly regular if and only if the relation*

(a) $$L\cap R=LAR$$

holds for every left ideal L and right ideal R of A.

Proof. For an associative ring A the following three conditions are equivalence with each other:

(1) A is strongly regular ring.

(2) $L\cap R=LR$ for every left ideal L and right ideal R of A.

(3) A is a two sided regular ring.

For the detail, see S. Lajos and F. Szász (1). We use the result above in our proof.

Necessity. Let A be a strongly regular ring. Then A satisfies the conditions (2) and (3). Let L be a left ideal and R be a right ideal of A. If a is any element of $L\cap R$, then there is an element x of A such that $a=axa$, so $a\in LAR$, i.e. $L\cap R\subset LAR$. Then $L\cap R\subset LAR \subset LR=L\cap R$, whence $L\cap R=LAR$.

Sufficiency. Let A be an associative ring satisfying the condition (a) for every left ideal L and right ideal R of A. If $R=A$ holds, then the relation (a) implies $L\cap A=LA^2$, whence every left ideal L of A is also a right ideal of A. Similarly, for $L=A$, (a) implies $A\cap R=A^2R$, whence every ideal R of A is a left ideal of A. Therefore A is a two sided ring. $L\cap R=LAR\subset LR\subset L\cap R$, so $L\cap R=LR$. Hence (a) implies (2). This means that A is a strongly regular ring.

Reference

[1] S. Lajos and F. Szász: Characterizations of strongly regular ring. Proc. Japan Acad., **46**, 38–40 (1970).