59. *Invariancy of Plancherel Measure under the Operation of Kronecker Product*

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1. Let G be a unimodular locally compact group of type I. For such a group, so-called Plancherel formula was given by F. I. Mautner [2], I. E. Segal [3], and H. Sunouchi [4], as follows.

Consider the dual $\hat{G}$ (the set of all equivalence classes of irreducible unitary representations) of G, and put $U_f(\omega)=\int_G f(g)U_g(\omega)dg$ for any function f in $L^1(G)$ and any unitary representation $\omega=\{\mathfrak{H}(\omega), U_g(\omega)\}$ of G. Then, there exists a measure μ (Plancherel measure) over $\hat{G}$, such that for any function f in $L^1(G)\cap L^2(G)$, the equation (1) is valid.

$$\|f\|^2=\int_{\hat{G}} |||U_f(\omega)|||^2\, d\mu(\omega). \tag{1}$$

Here $|||U_f(\omega)|||$ is the Hilbert-Schmidt norm of the operator $U_f(\omega)$.

This formula is considered as an extension of the Plancherel formula for abelian locally compact groups. But in this abelian case, $\hat{G}$ becomes an abelian locally compact group too, and the Plancherel measure μ is just invariant measure over $\hat{G}$.

The group operation of $\hat{G}$ is given by the ordinary product of characters as functions on G, that is, the Kronecker product of 1-dimensional representation. So the invariancy of Plancherel measure is that,

$$d\mu(\chi_0\otimes\chi)=d\mu(\chi), \qquad \text{for any } \chi_0 \text{ in } \hat{G}, \tag{2}$$

and this is equivalent to,

$$\int_{\hat{G}} |\tilde{f}(\chi_0\otimes\chi)|^2 d\mu(\chi)=\int_{\hat{G}} |\tilde{f}(\chi)|^2\, d\mu(\chi), \quad \text{for any } \chi_0 \text{ in } \hat{G} \text{ and } f \text{ in } L^1(G)\cap L^2(G). \tag{3}$$

Here $\tilde{f}$ shows the Fourier transform of f.

In general case, an analogue of (3) may be constructed as follows. At first, by virtue of (1), we replace Fourier transform $\tilde{f}$ of function f by the operator-valued function $U_f(\omega)$, then the term $|\tilde{f}(\chi_0\otimes\chi)|^2$ is replaced by $|||U_f(\omega_0\otimes\omega)|||^2$.

On the other hand, the well-known relation $\omega_0\otimes\mathfrak{R}\sim\sum_{\dim\omega_0}\oplus\mathfrak{R}$, for the regular representation $\mathfrak{R}$ and any representation ω_0, suggests that, in general form, the factor $(\dim\omega_0)^{-1}$ is needed in the left hand side. So, one of the purposes of this paper is to show the equation (4) for finite dimensional representation ω_0.