110. An Analogue of the Paley-Wiener Theorem for the Heisenberg Group

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1. Introduction. Let \( \mathbb{R} \) (resp \( \mathbb{C} \)) be the real (resp. complex) number field as usual. Let \( G \) be the \( n \)-th Heisenberg group, i.e. the group of all real matrices of the form
\[
\begin{pmatrix}
1 & a & c \\
0 & I_n & b \\
0 & 0 & 1
\end{pmatrix}
\]
where \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \), \( b = (b_1, \ldots, b_n) \in \mathbb{R}^n \), \( c \in \mathbb{R} \) and \( I_n \) is the identity matrix of \( n \)-th order. Let \( H \) be the abelian normal subgroup consisting of the elements of the form (1.1) with \( a = 0 \). For any real \( \eta \) we denote by \( \chi_\eta \) the unitary character of \( H \) defined by
\[
\chi_\eta: \begin{pmatrix}
1 & 0 & c \\
0 & I_n & b \\
0 & 0 & 1
\end{pmatrix}
\]
\[\rightarrow e^{2\pi i \eta c}.\]
Let \( U_\eta \) be the unitary representation of \( G \) induced by \( \chi_\eta \). Then the Plancherel theorem can be proved by means of \( U_\eta(\eta \in \mathbb{R}) \) (see e.g. [4]). However, as we have seen in the case of euclidean motion group ([2]), in order to prove an analogue of the Paley-Wiener theorem we have to consider the representations which have more parameters.

Let \( \check{H} \) be the dual group of \( H \). In this paper we consider the Fourier transform defined on \( \check{H} \equiv \mathbb{R}^{n+1} \).

Let \( C_c^\infty(G) \) be the set of all infinitely differentiable functions on \( G \) with compact support. For any \( \xi \in \mathbb{R}^n \) and \( \eta \in \mathbb{R} \) we denote by \( U^{\xi,\eta} \) the unitary representation of \( G \) induced by the unitary character \( \chi_{\xi,\eta} \) of
\[
H: \chi_{\xi,\eta}: \begin{pmatrix}
1 & 0 & c \\
0 & I_n & b \\
0 & 0 & 1
\end{pmatrix} = e^{2\pi i (\xi^t b) + 2\pi i \xi^t c}. \]
We define the (operator valued) Fourier transform \( T_f \) of \( f \in C_c^\infty(G) \) by
\[
T_f(\xi, \eta) = \int_G f(g)U^{\xi,\eta}_g dg,
\]
where \( dg \) is the Haar measure on \( G \). Then \( T_f(\xi, \eta) \) is an integral operator on \( L_2(\mathbb{R}^n) \) (§ 2). Denote by \( K_f(\xi, \eta; x, y) \) \((x, y \in \mathbb{R}^n)\) be the kernel function of \( T_f(\xi, \eta) \). We shall call \( K_f \) the scalar Fourier transform of \( f \).

The purpose of this paper is to determine the image of \( C_c^\infty(G) \) by the scalar Fourier transform (analogue of the Paley-Wiener theorem).