106. An Operator-Valued Stochastic Integral

By D. KANNAN*) and A. T. BHARUCHA-REID**)
Center for Research in Probability
Wayne State University, Detroit, Michigan 48202

(Comm. by Kinjirô KUNUGI, M. J. A., May 12, 1971)

1. Introduction. In this paper we define a stochastic integral of the form

$$\int_b^a \xi(t, \omega) dw(t, \omega) \tag{1}$$

where $\xi(t, \omega)$ is a second order Hilbert space-valued random function and $w(t, \omega)$ is a Hilbert space-valued Brownian motion or Wiener process. The stochastic integral to be defined is operator-valued; in particular, it is a function from a probability space into the space of Schmidt class operators on a Hilbert space. Hilbert space-valued stochastic integrals of operator-valued functions have been studied by several authors (cf., Mandrekar and Salehi [7], and Vakhaniya and Kandelski [10]). We first introduce some definitions and concepts which will be used in the development of the integral.

Let $(\Omega, \mathcal{A}, \mu)$ be a complete probability space, and let $\mathfrak{H}$ be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A mapping $x: \Omega \to \mathfrak{H}$ is said to be a *random element* in $\mathfrak{H}$, or an *$\mathfrak{H}$-valued random variable*, if for each $y \in \mathfrak{H}$, $\langle x(\omega), y \rangle$ is a real-valued random variable. Similarly, a mapping $L: \Omega \to \mathcal{B}(\mathfrak{H})$ (where $\mathcal{B}(\mathfrak{H})$ is the Banach algebra of endomorphisms of $\mathfrak{H}$) is said to be a *random operator* if, for every $x, y \in \mathfrak{H}$, $\langle L(\omega)x, y \rangle$ is a real-valued random variable.

Let x and y be two given elements in $\mathfrak{H}$. The tensor product of x and y, written $x \otimes y$, is an endomorphism in $\mathfrak{H}$ whose defining equation is $(x \otimes y)h = \langle h, y \rangle x$, $h \in \mathfrak{H}$. A simple consequence of this definition is $(x_1 \otimes y_1)(x_2 \otimes y_2) = \langle x_2, y_1 \rangle (x_1 \otimes y_2)$. We refer to Schattan [8] for a discussion of the operator $x \otimes y$ and its properties. Now let $x(\omega)$ and $y(\omega)$ be two $\mathfrak{H}$-valued random variables; and consider the tensor product $x(\omega) \otimes y(\omega)$. Falb [3] (cf. also [5]) has shown that the operator-valued function $x(\omega) \otimes y(\omega)$ is measurable; i.e., it is a random operator. Falb established the measurability of $x(\omega) \otimes y(\omega)$ using open sets; however, it follows easily from the definitions of a random operator and the tensor product operator.

An $\mathfrak{H}$-valued random function $\{w(t, \omega), t \in [a, b]\}$ is said to be a

*) Present address: Department of Mathematics, New York University, University Heights, Bronx, New York 10453.

**) Research supported by National Science Foundation Grant No. GP-13741.