99. *Note on Simple Semigroups*

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1. By a *left* (*right*) *ideal* of a semigroup S we mean a non-empty subset X of S such that $SX\subseteq X$ ($XS\subseteq X$). By a *two-sided ideal*, or simply *ideal*, we mean a subset of S which is both a left and a right ideal of S. A semigroup S is called *simple* if it contains no proper two-sided ideal. We denote by $[x]$ the principal ideal of S generated by x of S. A semigroup S is called *left* (*right*) *zero* if $xy=x$ ($xy=y$) for every $x, y \in S$. Let $\mathfrak{T}(S)$ be the set of all non-empty subsets of a semigroup S. A binary operation is defined in $\mathfrak{T}(S)$ as follows: For X, $Y \in \mathfrak{T}(S)$,

$$XY=\{xy\,;\, x\in X, y\in Y\}.$$

Then it is easily seen that $\mathfrak{T}(S)$ is a semigroup.

Let $\mathfrak{J}(S)$ be the set of all ideals of a semigroup S and $\mathfrak{P}(S)$ the set of all principal ideals of S. It is clear that $\mathfrak{J}(S)$ is a subsemigroup of $\mathfrak{T}(S)$. The author proved in [2] that $\mathfrak{J}(S)$ is an idempotent semigroup if and only if $\mathfrak{P}(S)$ is an idempotent semigroup, and then both $\mathfrak{J}(S)$ and $\mathfrak{P}(S)$ are commutative. In this note we shall prove the following theorem:

Theorem 1. *Let S be a semigroup. Then S is a simple semigroup if and only if any one of the following conditions* (A)–(D) *holds*:

(A) $\mathfrak{J}(S)$ *is a left zero semigroup.*

(B) $\mathfrak{J}(S)$ *is a right zero semigroup.*

(C) $\mathfrak{P}(S)$ *is a left zero semigroup.*

(D) $\mathfrak{P}(S)$ *is a right zero semigroup.*

2. First we mention a result from our previous paper [2].

Lemma 2. *The following statements on a semigroup S are equivalent*:

(i) $X^2=X$ *for every* $X\in\mathfrak{J}(S)$.

(ii) $X\cap Y=XY$ *for every* $X, Y\in\mathfrak{J}(S)$.

(iii) $[x]^2=[x]$ *for every* $[x]\in\mathfrak{P}(S)$.

(iv) $[x]\cap[y]=[x][y]$ *for every* $[x], [y]\in\mathfrak{P}(S)$.

3. Proof of Theorem 1. Assume that S is simple, then it is clear that (A) holds. Conversely, if (A) holds, then, since $\mathfrak{J}(S)$ is an idempotent semigroup, it follows from (i), (ii) of Lemma 2 that

$$X=XY=X\cap Y$$