169. *Some Dual Properties on Modules*

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As mutually dual notions on modules, we have projective module and injective module, quasi-projective module and quasi-injective module, minimal homomorphism and essential homomorphism etc. We know that some properties for ones have dual properties for others. But it is not necessarily always true. In this short note, we collect some properties on modules which has dual property. Some dual property of one can be easily proved by exchanging the notions to its duals. We assume that R is any ring with unit element and every R-module is unitary R-right (or left) module. For R-module M and its R-sub module N, the symbol $N \hookrightarrow M$ denotes the inclusion map $x \mapsto x$, and $M \xrightarrow{-} M/N$ denotes the canonical epimorphism $x \mapsto \bar{x}$.

Property 1. *For any R-module M, $\mathfrak{A} = \{f \in End_R(M)\,;\, Ker\, f \hookrightarrow M$ is essential$\}$ is a two sided ideal of $End_R(M)$.*

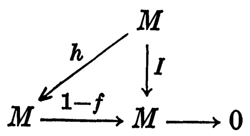
Proof. For any $f, f' \in \mathfrak{A}$, $Ker\, f \cap Ker\, f' \subset Ker\,(f+f') \subset M$, therefore $f+f' \in \mathfrak{A}$. For $f \in \mathfrak{A}$ and $g \in End_R(M)$, $M \supset Ker\,(gf)\; Ker\, f$, hence $gf \in \mathfrak{A}$. On the other hand, for any R-submodule U of M $Ker\, f \cap g(U) \subset g\,(Ker\,(fg) \cap U)$, hence if $Ker\,(fg) \cap U = 0$ then $Ker\, f \cap g(U) = 0$ and so $g(U) = 0$. Therefore $U \subset Ker\, g \subset Ker\,(f \cdot g)$ and $U \subset Ker\,(f \cdot g) \cap U = 0$. Accordingly $f \cdot g \in \mathfrak{A}$.

Property 1*) (Harada and Kambara [1], Lemma 1). *For any R-module M, $\mathfrak{A}^* = \{f \in End_R(M)\,;\, M \xrightarrow{-} Coker\, f$ is minimal$\}$ is a twosided ideal of R.*

Property 2 (Faith [2], p. 44, Theorem 1). *If M is quasi-injective R-module, then is Jacobson radical of $End_R(M)$.*

Property 2*). *If M is quasi-projective R-module, then $\mathfrak{A}^*$ is Jacobson radical of $End_R(M)$.*

Proof. Let J be the radical of $End_R(M)$. Then $J \supset \mathfrak{A}^*$ is obvious. Because, for any f in $\mathfrak{A}^*$, $Im\, f + Im\,(1-f) = M$, therefore $Im\,(1-f) = M$, i.e. $1-f : M \to M$ is R-epimorphism. By quasi-projective of M, $1-f$ has right inverse h in $End_R(M)$, i.e.



is commutative. Therefore $f \in J$. Conversely, let f be in J, and U