20. On Exponential Semigroups. I[1]

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1. Introduction. By a semilattice-congruence ξ on a semigroup S we mean a congruence on S such that S/ξ is a semilattice, i.e., a commutative idempotent semigroup. As is well known there is a smallest semilattice-congruence ρ on S in the sense that ρ is contained in all semilattice-congruences ξ on S. If $\rho = S\times S$, then S is called semilattice-indecomposable.

A semigroup is called medial if it satisfies the identity $xyzu = xzyu$; a semigroup A is called archimedian if for every $a, b \in A$ there are elements x, y, z, u of A^1 and positive integers m, n such that $xay = b^m$ and $zbu = a^n$. Chrislock [1, 2] proved the following:

Theorem 1. *Every medial semigroup S is a semilattice of archimedean semigroups, and the congruence relation ρ on S which induces this decomposition is a smallest semilattice-congruence on S and ρ is given by $a\rho b$ if and only if $xay = b^m$ and $zbu = a^n$ for some $x, y, z, u \in S^1$ and some $m, n > 0$. A medial semigroup is semilattice-indecomposable if and only if it is archimedean.*

Theorem 2. *Let S be a medial semigroup. S is an archimedean semigroup with idempotent if and only if S is an ideal extension of the direct product I of an abelian group G and a rectangular band B by the medial nil-semigroup N.*

The above Theorems 1, 2 were generalizations of Kimura and Tamura's result for commutative semigroups [5]. The purpose of this note is to extend Chrislock's results to the exponential case. Furthermore the authors intend the development of Theorem 2 not only in the medial case, but in the exponential case. In Theorem 2 mediality is assumed in the "if" part and "only if" part. If mediality is assumed in the "only if" part, then the "only if" part is still true, but the "if" part is not true in general, that is to say, an ideal extension of $I = G\times B$ by medial N need not be medial. Part I of this paper establishes the extension of Theorem 1 and the "only if" part of Theorem 2. The "if" part will be discussed in Part II.

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